

# EVALUATION OF BINOMIAL DOUBLE SUMS INVOLVING ABSOLUTE VALUES

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ABSTRACT. We show that double sums of the form

$$\sum_{i,j=-n}^n |i^s j^t (i^k - j^k)^\beta| \binom{2n}{n+i} \binom{2n}{n+j}$$

can always be expressed in terms of a linear combination of just four functions, namely  $\binom{4n}{2n}$ ,  $\binom{2n}{n}^2$ ,  $4^n \binom{2n}{n}$ , and  $16^n$ , with coefficients that are rational in  $n$ . We provide two different proofs: one is algorithmic and uses the second author's computer algebra package **Sigma**; the second is based on complex contour integrals.

## 1. INTRODUCTION

Motivated by work in [6] concerning the Hadamard maximal determinant problem [9], Brent and Osborn [5] proved the double sum evaluation

$$\sum_{i,j=-n}^n |i^2 - j^2| \binom{2n}{n+i} \binom{2n}{n+j} = 2n^2 \binom{2n}{n}^2. \quad (1.1)$$

It should be noted that the difficulty in evaluating this sum lies in the appearance of the absolute value. Without the absolute value, the summation could be carried out straightforwardly by means of the binomial theorem. Together with Ohtsuka and Prodinger, they went on in [3] (see [4] for the published version) to consider more general double sums of the form

$$\sum_{i,j=-n}^n |i^s j^t (i^k - j^k)^\beta| \binom{2n}{n+i} \binom{2n}{n+j}, \quad (1.2)$$

mostly for small positive integers  $k, s, t, \beta$ . In several cases, they found explicit evaluations of such sums — sometimes with proof, sometimes conjecturally.

The purpose of the current paper is to provide a complete treatment of double sums of the form (1.2). More precisely, using the computer algebra package **Sigma** [13], we were led to the conjecture that these double sums can always be expressed in terms of a linear

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combination of just four functions, namely  $\binom{4n}{2n}$ ,  $\binom{2n}{n}^2$ ,  $4^n \binom{2n}{n}$ , and  $16^n$ , with coefficients that are rational in  $n$ . We demonstrate this observation in Theorems 13 and 14, in a much more precise form.

It is not difficult to see that the problem of evaluation of double sums of the form (1.2) can be reduced to the evaluation of sums of the form

$$\sum_{0 \leq i \leq j} i^s j^t \binom{2n}{n+i} \binom{2n}{n+j} \quad (1.3)$$

(and a few simpler *single* sums). See the proofs of Theorems 13 and 14 in Section 6 and Remark 15(1). We furthermore show (see the proof of Proposition 10 in Section 5, which may be considered as the actual main result of the present paper) that for the evaluation of double sums of the form (1.3) it suffices to evaluate four *fundamental* double sums, given in Lemmas 1–4 in Section 2. We provide two different proofs, one using computer algebra, and one using complex contour integrals. We believe that both proofs are of intrinsic interest. The algorithmic proof is described in Section 3. There, we explain that the computer algebra package **Sigma** can be used in a completely automatic fashion to evaluate double sums of the form (1.3). In particular, the reader can see how we empirically discovered our main results in Sections 5 and 6. The second proof, based on the power of complex integration, is explained in Section 4.

We close our paper by establishing another conjecture from [3, Conj. 3.1], namely the inequality (see Theorem 16 in Section 7)

$$\sum_{i,j} |j^2 - i^2| \binom{2n}{n+i} \binom{2m}{m+j} \geq 2mn \binom{2m}{m} \binom{2n}{n}.$$

It should be noted that the double sum on the left-hand side generalises the one on the left-hand side of (1.1) in that the binomial coefficient  $\binom{2n}{n+j}$  gets replaced by  $\binom{2m}{m+j}$ . We show moreover that equality holds if and only if  $m = n$ , in which case the evaluation (1.1) applies. Although Lemmas 1–4 would provide a good starting point for a proof of the inequality, we prefer to use a more direct approach, involving an application of Gosper’s algorithm [7] at a crucial point.

We wish to point out that Bostan, Lairez and Salvy [1] have developed an algorithmic approach — based on contour integrals — that is capable of automatically finding a recurrence for the double sum (1.2) for any particular choice of  $s, t, k, \beta$ , and, thus, is able to establish an evaluation of such a sum (such as (1.1), for example) once the right-hand side is found.

Our final remark is that some of the double sums (1.2) can be embedded into infinite families of multidimensional sums that still allow for closed form evaluations, see [2].

## 2. THE FUNDAMENTAL LEMMAS

In this section, we state the summation identities which form the basis of the evaluation of double sums of the form (1.3). We provide two different proofs, the first being algorithmic — see Section 3, the second making use of complex integration — see Section 4.

**Lemma 1.** *For all non-negative integers  $n$  and  $m$ , we have*

$$\begin{aligned} \sum_{0 \leq i \leq j} \binom{2n}{n+i} \binom{2m}{m+j} &= 2^{2n+2m-3} + \frac{1}{4} \binom{2n+2m}{n+m} + \frac{1}{2} \binom{2n}{n} \binom{2m}{m} \\ &\quad + 2^{2m-2} \binom{2n}{n} - \frac{1}{8} \sum_{\ell=0}^{n-m} \binom{2n-2\ell}{n-\ell} \binom{2m+2\ell}{m+\ell}, \end{aligned} \quad (2.1)$$

where the sum has to be interpreted as explained in Lemma 9.

**Lemma 2.** *For all non-negative integers  $n$  and  $m$ , we have*

$$\begin{aligned} \sum_{0 \leq i \leq j} i \binom{2n}{n+i} \binom{2m}{m+j} &= -\frac{n}{4} \binom{2n+2m}{n+m} + n 2^{2m-2} \binom{2n}{n} \\ &\quad + \frac{n}{8} \sum_{\ell=0}^{n-m} \binom{2n-2\ell}{n-\ell} \binom{2m+2\ell}{m+\ell} - \frac{n}{2} \sum_{\ell=0}^{n-m-1} \binom{2n-2\ell-2}{n-\ell-1} \binom{2m+2\ell}{m+\ell}, \end{aligned} \quad (2.2)$$

where the sums have to be interpreted as explained in Lemma 9.

**Lemma 3.** *For all non-negative integers  $n$  and  $m$ , we have*

$$\begin{aligned} \sum_{0 \leq i \leq j} j \binom{2n}{n+i} \binom{2m}{m+j} &= \frac{m}{4} \binom{2n+2m}{n+m} + m \binom{2n}{n} \binom{2m-2}{m-2} \\ &\quad - \frac{m}{8} \sum_{\ell=0}^{n-m} \binom{2n-2\ell}{n-\ell} \binom{2m+2\ell}{m+\ell} + \frac{m}{2} \sum_{\ell=0}^{n-m+1} \binom{2n-2\ell}{n-\ell} \binom{2m+2\ell-2}{m+\ell-1}, \end{aligned} \quad (2.3)$$

where the sum has to be interpreted as explained in Lemma 9.

**Lemma 4.** *For all non-negative integers  $n$  and  $m$ , we have*

$$\begin{aligned} \sum_{0 \leq i \leq j} i j \binom{2n}{n+i} \binom{2m}{m+j} &= \frac{mn}{2} \binom{2n+2m-2}{n+m-1} - \frac{mn}{2} \binom{2n+2m-2}{n+m-2} \\ &\quad + \frac{mn}{8} \sum_{\ell=0}^{n-m} \binom{2n-2\ell}{n-\ell} \binom{2m+2\ell}{m+\ell} - \frac{mn}{2} \sum_{\ell=0}^{n-m-1} \binom{2n-2\ell-2}{n-\ell-1} \binom{2m+2\ell}{m+\ell}, \end{aligned} \quad (2.4)$$

where the sum has to be interpreted as explained in Lemma 9.

### 3. PROOF OF LEMMAS 1–4 USING THE COMPUTER ALGEBRA PACKAGE **Sigma**

We restrict our attention to the proof of Lemma 1. Algorithmic proofs of Lemmas 2–4 can be obtained completely analogously and are therefore omitted for the sake of brevity.

We seek an alternative representation of the double sum

$$S(n, m) = \sum_{0 \leq i \leq j} \binom{2n}{n+i} \binom{2m}{m+j} \quad (3.1)$$

for all non-negative integers  $m, n$  with the following property: if one specialises  $m$  (respectively  $n$ ) to a non-negative integer or if one knows the distance between  $n$  and  $m$ , then the evaluation of the double sum should be performed in a direct and simple fashion. In order to accomplish this task, we utilise the summation package **Sigma** [13].

The sum (3.1) can be rewritten in the form

$$S(n, m) = \sum_{j=0}^m f(n, m, j) \quad (3.2)$$

with

$$f(n, m, j) = \binom{2m}{j+m} \sum_{i=0}^j \binom{2n}{i+n}. \quad (3.3)$$

Given this sum representation we will exploit the following summation spiral that is built into **Sigma**:

- (1) Calculate a linear recurrence in  $m$  of order  $d$  (for an appropriate positive integer  $d$ ) for the sum  $S(n, m)$  by the creative telescoping paradigm;
- (2) solve the recurrence in terms of (indefinite) nested sums over hypergeometric products with respect to  $m$  (the corresponding sequences are also called *d'Alembertian solutions*);
- (3) combine the solutions into an expression  $\text{RHS}(n, m)$  such that  $S(n, l) = \text{RHS}(n, l)$  holds for all  $n$  and  $l = 0, 1, \dots, d-1$ .

Then this implies that  $S(n, m) = \text{RHS}(n, m)$  holds for all non-negative integers  $m, n$ .

*Remark 5.* This summation engine can be considered as a generalization of [12] that works not only for hypergeometric products but for expressions in terms of nested sums over such hypergeometric products. It is based on a constructive summation theory of difference rings and fields [15, 16] that enhances Karr's summation approach [10] in various directions.

In the following paragraphs, we assume that  $m \leq n$ . We activate **Sigma**'s summation spiral.

**STEP 1.** Observe that our sum (3.2) with summand given in (3.3) is already in the right input form for **Sigma**: the summation objects of (3.3) are given in terms of nested sums over hypergeometric products. More precisely, let  $\mathcal{S}_j$  denote the shift operator with respect to  $j$ , that is,  $\mathcal{S}_j F(j) := F(j+1)$ . Then, if one applies this shift operator to the arising objects of  $f(n, m, j)$ , one can rewrite them again in their non-shifted versions:

$$\begin{aligned} \mathcal{S}_j \binom{2m}{j+m} &= \frac{m-j}{1+j+m} \binom{2m}{j+m}, \\ \mathcal{S}_j \sum_{i=0}^j \binom{2n}{i+n} &= \sum_{i=0}^j \binom{2n}{i+n} + \frac{n-j}{1+j+n} \binom{2n}{j+n}. \end{aligned} \quad (3.4)$$

With the help of these identities, we can look straightforwardly for a linear recurrence in the free integer parameter  $m$  as follows. First, we load **Sigma** into the computer algebra system *Mathematica*,

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

and enter our definite sum  $S(n, m)$ :

In[2]:= **mySum** = **SigmaSum**[**Binomial**[**2m**, **j** + **m**]**SigmaSum**[**Binomial**[**2n**, **i** + **n**], {**i**, **0**, **j**}], {**j**, **0**, **m**}]

Out[2]= 
$$\sum_{j=0}^m \binom{2m}{j+m} \sum_{i=0}^j \binom{2n}{i+n}$$

Then we compute a recurrence in  $m$  by executing the function call

In[3]:= **rec** = **GenerateRecurrence**[**mySum**, **m**][[1]]

Out[3]= 
$$\text{SUM}[m+1] - 4\text{SUM}[m] == -\frac{1}{1+m+n} \sum_{i=0}^m \binom{2m}{i+m} \binom{2n}{i+n} + \frac{mn}{(m+1)(1+m+n)} \binom{2m}{m} \binom{2n}{n}$$

This means that  $\text{SUM}[m] = S(n, m)(= \text{mySum})$  is a solution of the output recurrence. But what is going on behind the scenes? Roughly speaking, Zeilberger's creative telescoping paradigm [12] is carried out in the setting of difference rings. More precisely, one tries to compute a recurrence for the summand  $f(n, m, j)$  of the form

$$c_0(n, m)f(n, m, j) + c_1(n, m)f(n, m+1, j) + \cdots + c_d(n, m)f(n, m+d, j) = g(n, m, j+1) - g(n, m, j), \quad (3.5)$$

for  $d = 0, 1, 2, \dots$ . In our particular instance, **Sigma** is successful for  $d = 1$  and delivers the solution  $c_0(n, m) = -4$ ,  $c_1(n, m) = 1$ , and

$$g(n, m, j) = \frac{(2j-1)}{-1+j-m} \binom{2m}{j+m} \sum_{i=0}^j \binom{2n}{i+n} + \frac{j-n}{1+m+n} \binom{2m}{j+m} \binom{2n}{j+n} + \frac{1}{-1-m-n} \sum_{i=0}^j \binom{2m}{i+m} \binom{2n}{i+n}, \quad (3.6)$$

which holds for all non-negative integers  $j, m, n$  with  $0 \leq j \leq m \leq n$ . The correctness can be verified by substituting the right-hand side of (3.3) into (3.5), rewriting the summation objects in terms of  $\binom{2m}{j+m}$  and  $\sum_{i=0}^j \binom{2n}{i+n}$  using the relations given in (3.4) and  $\mathcal{S}_m \binom{2m}{j+m} = \frac{2(m+1)(2m+1)}{(m-j+1)(1+j+m)} \binom{2m}{j+m}$ , and applying simple rational function arithmetic. We recall that we assumed  $m \leq n$ , and this restriction is indeed essential for being allowed to use **Sigma** in the described setup. However, the above check reveals that the result is in fact correct without any restriction on the relative sizes of  $m$  and  $n$ .

Finally, by summing (3.5) over  $j$  from 0 to  $m$ , we obtain the linear recurrence

$$\sum_{j=0}^m f(n, m+1, j) - 4 \sum_{j=0}^m f(n, m, j) = - \sum_{j=0}^{m+1} \binom{2n}{i+n} + \frac{1}{-1-m-n} \sum_{i=0}^m \binom{2m}{i+m} \binom{2n}{i+n} + \frac{mn}{(m+1)(1+m+n)} \binom{2m}{m} \binom{2n}{n}.$$

which, by the above remark, holds for all non-negative integers  $m, n$ . As is straightforward to see, this is indeed equivalent to **Out**[3].

**STEP 2.** We now apply our summation toolbox to the definite sum  $\sum_{i=0}^m \binom{2m}{i+m} \binom{2n}{i+n}$  and obtain

$$\sum_{i=0}^m \binom{2m}{m+i} \binom{2n}{n+i} = \frac{1}{2} \binom{2m}{m} \binom{2n}{n} + \frac{1}{2} \binom{2m+2n}{m+n}. \quad (3.7)$$

Note that the calculations can be verified rigorously and as a consequence we obtain a proof that the identity holds for all non-negative integers  $m, n$ . Since we remain in this particular case purely in the hypergeometric world, one could also use the classical toolbox described in [12]. Yet another (classical) proof consists in observing that the sum on the left-hand side of (3.7) can be rewritten as

$$\begin{aligned} \frac{1}{2} \left( \sum_{i=0}^m \binom{2m}{m+i} \binom{2n}{n-i} + \sum_{i=0}^m \binom{2m}{m-i} \binom{2n}{n+i} \right) \\ = \frac{1}{2} \left( \sum_{i=0}^{2m} \binom{2m}{i} \binom{2n}{n+m-i} + \binom{2m}{m} \binom{2n}{n} \right), \end{aligned}$$

and then evaluating the sum on the right-hand side by means of the Chu–Vandermonde summation formula.

As a consequence, we arrive at the linear recurrence

$$\begin{aligned} \text{In}[4] := \text{rec} = \text{rec} /. \sum_{i=0}^m \binom{2m}{i+m} \binom{2n}{i+n} \rightarrow \frac{1}{2} \binom{2m}{m} \binom{2n}{n} + \frac{1}{2} \binom{2m+2n}{m+n} \\ \text{Out}[4] = \text{SUM}[m+1] - 4\text{SUM}[m] == -\frac{\binom{2m+2n}{m+n}}{1+m+n} \frac{1}{2} + \frac{(-1-m+2mn)\binom{2m}{m}\binom{2n}{n}}{2(m+1)(1+m+n)} \end{aligned}$$

Now we can activate **Sigma**'s recurrence solver with the function call

$$\begin{aligned} \text{In}[5] := \text{recSol} = \text{SolveRecurrence}[\text{rec}, \text{SUM}[m]] \\ \text{Out}[5] = \{ \{0, 2^{2m}\}, \{1, \frac{1}{4} \binom{2m}{m} \binom{2n}{n} + \frac{1}{4} \binom{2m+2n}{m+n} + 2^{2m} \binom{2n}{n} \left( -\frac{1}{4} + \frac{1}{4} n \sum_{i=0}^m \frac{2^{-2i} \binom{2i}{i}}{i+n} \right) \} \} \end{aligned}$$

This means that the first entry of the output is the solution of the homogeneous version of the recurrence, and the second entry is a solution of the recurrence itself. Hence, the general solution is

$$c 2^{2m} + \frac{1}{4} \binom{2m}{m} \binom{2n}{n} + \frac{1}{4} \binom{2m+2n}{m+n} + 2^{2m} \binom{2n}{n} \left( -\frac{1}{4} + \frac{1}{4} n \sum_{i=0}^m \frac{2^{-2i} \binom{2i}{i}}{i+n} \right), \quad (3.8)$$

where the constant  $c$  (free of  $m$ ) can be freely chosen. We note that this solution can be easily verified by substituting it into **rec** computed in **Out[4]** and using the relations

$$\begin{aligned} \mathcal{S}_m \binom{2m}{m} &= \frac{2(2m+1)}{m+1} \binom{2m}{m}, \quad \mathcal{S}_m \binom{2m+2n}{m+n} = \frac{2(2m+2n+1)}{m+n+1} \binom{2m+2n}{m+n}, \\ \mathcal{S}_m \sum_{i=0}^m \frac{2^{-2i} \binom{2i}{i}}{i+n} &= \sum_{i=0}^m \frac{2^{-2i} \binom{2i}{i}}{i+n} + \frac{2^{-2m}(2m+1)}{2(m+1)(1+m+n)} \binom{2m}{m}. \end{aligned}$$

STEP 3. Looking at the initial value  $S(n, 0) = \binom{2n}{n}$ , we conclude that the specialization  $c = \frac{1}{2} \binom{2n}{n}$  in (3.8) equals  $S(n, m)$  for all  $n \geq 0$  and  $m = 0$ .

Summarising, we have found (together with a proof) the representation

$$S(n, m) = 2^{2m-2} \binom{2n}{n} n \sum_{i=0}^m \frac{2^{-2i} \binom{2i}{i}}{i+n} + 2^{2m-2} \binom{2n}{n} + \frac{1}{4} \binom{2m}{m} \binom{2n}{n} + \frac{1}{4} \binom{2m+2n}{m+n}, \quad (3.9)$$

which holds for all non-negative integers  $m, n$ . This last calculation step can be also carried out within **Sigma**, by making use of the function call

$$\begin{aligned} \text{In[6]} &:= \text{FindLinearCombination}[\text{recSol}, \{0, \{\binom{2n}{n}\}\}, m, 1] \\ \text{Out[6]} &= 2^{2m-2} \binom{2n}{n} n \sum_{i=0}^m \frac{2^{-2i} \binom{2i}{i}}{i+n} + 2^{2m-2} \binom{2n}{n} + \frac{1}{4} \binom{2m}{m} \binom{2n}{n} + \frac{1}{4} \binom{2m+2n}{m+n} \end{aligned}$$

Strictly speaking, the above derivations contained one “human” (= non-automatic) step, namely at the point where we checked (3.6) and observed that this relation actually holds without the restriction  $m \leq n$ . For the algorithmic “purist” we point out that it is also possible to set up the problem appropriately under the restriction  $m > n$  (by splitting the double sum  $S(n, m)$  into two parts) so that **Sigma** is applicable. Not surprisingly, **Sigma** finds (3.9) again.

In this article, we are particularly interested in the evaluation of  $S(n, m)$  if one fixes the distance  $r = n - m \geq 0$  (or  $r = m - n \geq 0$ ). In order to find such a representation for the case  $m \leq n$ , we manipulate the obtained sum

$$\sum_{i=0}^m \frac{2^{-2i} \binom{2i}{i}}{i+n} = \sum_{i=0}^m \frac{2^{-2i} \binom{2i}{i}}{i+r+m} := T(m, r) \quad (3.10)$$

in (3.9) further by applying once more **Sigma**’s summation spiral (where  $r$  takes over the role of  $m$ ).

STEP 1. Using **Sigma** (alternatively one could use the Paule and Schorn implementation [11] of Zeilberger’s algorithm), we obtain the recurrence

$$2(m+r)T(m, r) + (-1 - 2m - 2r)T(m, r+1) = \frac{2^{-2m}(2m+1)\binom{2m}{m}}{2m+r+1}.$$

STEP 2. Using **Sigma**'s recurrence solver we obtain the general solution

$$d \frac{2^{2r} m \binom{2m}{m}}{\binom{2m+2r}{m+r}(m+r)} + \frac{2^{-2m} \binom{2m}{m}}{m+r} - \frac{2^{-2m+2r} (4m+1) \binom{2m}{m}^2}{2 \binom{2m+2r}{m+r}(m+r)} - \frac{2^{2r-2m} m \binom{2m}{m}}{\binom{2m+2r}{m+r}(m+r)} \sum_{i_1=0}^r \frac{2^{-2i_1} \binom{2m+2i_1}{m+i_1}}{2m+i_1},$$

where the constant  $d$  (free of  $r$ ) can be freely chosen.

STEP 3. Looking at the initial value

$$T(m, 0) = \sum_{i=0}^m \frac{2^{-2i} \binom{2i}{i}}{i+m} = \frac{2^{2m-1}}{m \binom{2m}{m}} + \frac{2^{-2m-1} \binom{2m}{m}}{m},$$

which we simplified by another round of **Sigma**'s summation spiral, we conclude that we have to specialise  $d$  to

$$d = \frac{2^{2m-1}}{m \binom{2m}{m}} + \frac{2^{-2m-1} (4m+1) \binom{2m}{m}}{m}.$$

With this choice, we end up at the identity

$$T(m, r) = -\frac{2^{2r-2m} m \binom{2m}{m}}{\binom{2m+2r}{m+r}(m+r)} \sum_{i=0}^r \frac{2^{-2i} \binom{2i+2m}{i+m}}{i+2m} + \frac{2^{-2m} \binom{2m}{m}}{m+r} + \frac{2^{2m+2r-1}}{\binom{2m+2r}{m+r}(m+r)},$$

being valid for all non-negative integers  $r, m$ . Finally, performing the substitution  $r \rightarrow n - m$ , we find the identity

$$T(m, n) = -\frac{2^{2n-4m} \binom{2m}{m}}{n \binom{2n}{n}} m \sum_{i=0}^{n-m} \frac{2^{-2i} \binom{2i+2m}{i+m}}{i+2m} + \frac{2^{2n-1}}{n \binom{2n}{n}} + \frac{2^{-2m} \binom{2m}{m}}{n}, \quad (3.11)$$

which holds for all non-negative integers  $n, m$  with  $n \geq m$ . By substituting this result into (3.9), we see that we have discovered *and* proven that

$$\begin{aligned} S(n, m) &= -2^{-2m+2n-2} \binom{2m}{m} m \sum_{i=0}^{n-m} \frac{2^{-2i} \binom{2i+2m}{i+m}}{i+2m} \\ &\quad + 2^{2m-2} \binom{2n}{n} + \frac{1}{2} \binom{2m}{m} \binom{2n}{n} + \frac{1}{4} \binom{2m+2n}{m+n} + 2^{2m+2n-3}, \end{aligned} \quad (3.12)$$

which is valid for all non-negative integers  $n, m$  with  $n \geq m$ . In a similar fashion, if  $m \geq n$ , we obtain

$$\begin{aligned} S(n, m) &= 2^{2m-2n-2} \binom{2n}{n} n \sum_{i=0}^{m-n} \frac{2^{-2i} \binom{2i+2n}{i+n}}{i+2n} \\ &\quad + 2^{2m-2} \binom{2n}{n} + \frac{1}{4} \binom{2m}{m} \binom{2n}{n} + \frac{1}{4} \binom{2m+2n}{m+n} + 2^{2m+2n-3}. \end{aligned} \quad (3.13)$$

We note that the interaction of the summation Steps 1–3 is carried out at various places in a recursive manner. In order to free the user from all these mechanical but rather



subtle calculation steps, the additional package `EvaluateMultiSums` [14] has been developed recently. It coordinates all these calculation steps cleverly and discovers identities as above completely automatically whenever such a simplification in terms of nested sums over hypergeometric products is possible. For instance, after loading the package

`ln[7]:= << EvaluateMultiSum.m`

`EvaluateMultiSums by Carsten Schneider © RISC-Linz`

we can transform the sum (3.1) into the desired form by executing the function call

`ln[8]:= res = EvaluateMultiSum[( $\frac{2n}{n+i}$ )( $\frac{2m}{m+j}$ ), {{i, 0, j}}, {j, 0, m}, {m, n}, {0, 0}, {n, ∞}]`

$$\text{Out[8]} = \frac{(2n+1)2^{2m-3}(2n)!}{n^2((n-1)!)^2} \sum_{i=1}^m \frac{2^{-2i} \binom{2i}{i}}{1+i+n} + \frac{(4n+3)2^{2m-3}(2n)!}{n^2(n+1)((n-1)!)^2} + \frac{(3+4m+2n)\binom{2m}{m}(2n)!}{8n^2(1+m+n)((n-1)!)^2} + \frac{(2m+2n)!}{4n^2((n-1)!)^2((n+1)_n)^2}$$

Here, `Sigma` uses the *Pochhammer symbol*  $(\alpha)_m$  defined by

$$(\alpha)_m = \begin{cases} \alpha(\alpha+1)(\alpha+2) \cdots (\alpha+m-1), & \text{for } m > 0, \\ 1, & \text{for } m = 0, \\ 1/(\alpha-1)(\alpha-2)(\alpha-3) \cdots (\alpha+m), & \text{for } m < 0, \end{cases} \quad (3.14)$$

of which we shall also make use later frequently. The parameters  $m, n$  in the calculation above are bounded from below by 0, 0 and from above by  $n, \infty$ , respectively. If one prefers a representation purely in terms of binomial coefficients, one may execute the following function calls:

`ln[9]:= res = SigmaReduce[res, m, Tower → {( $\frac{2m}{m}$ ), ( $\frac{2n+2m}{n+m}$ )}];`

`ln[10]:= res = SigmaReduce[res, n, Tower → {( $\frac{2n}{n}$ )}];`

$$\text{Out[10]} = 2^{2m-3}(2n+1) \binom{2n}{n} \sum_{i=1}^m \frac{2^{-2i} \binom{2i}{i}}{1+i+n} + \frac{(4n+3)2^{2m-3} \binom{2n}{n}}{n+1} + \frac{(3+4m+2n)\binom{2m}{m}\binom{2n}{n}}{8(1+m+n)} + \frac{1}{4} \binom{2m+2n}{m+n}$$

If one rewrites the arising sum manually by means of the function call below, one finally ends up exactly at the result given in (3.9):

`ln[11]:= res = SigmaReduce[res, m, Tower → { $\sum_{i=1}^m \frac{2^{-2i} \binom{2i}{i}}{i+n}$ }]`

$$\text{Out[11]} = 2^{2m-2} \binom{2n}{n} \sum_{i=1}^m \frac{2^{-2i} \binom{2i}{i}}{i+n} + 2^{2m-1} \binom{2n}{n} + \frac{1}{4} \binom{2m}{m} \binom{2n}{n} + \frac{1}{4} \binom{2m+2n}{m+n}$$

Analogously one can carry out these calculation steps to calculate the simplification given in (3.11) automatically.

Comparison with Lemma 1 reveals that (3.12) or (3.13) do not quite agree with the right-hand side of (2.1). For example, in order to prove that (3.12) is equivalent with (2.1), we would have to establish the identity

$$\frac{1}{8} \sum_{l=0}^{n-m} \binom{2m+2l}{m+l} \binom{2n-2l}{n-l} = 2^{-2m+2n-2} \binom{2m}{m} m \sum_{i=0}^{n-m} \frac{2^{-2i} \binom{2i+2m}{i+m}}{i+2m}.$$

This can, of course, be routinely achieved by using the Paule and Schorn [11] implementation [11] of Zeilberger's algorithm. Alternatively, we may use our **Sigma** summation technology again. Let

$$T'(n, m) := \sum_{l=0}^{n-m} \binom{2m+2l}{m+l} \binom{2n-2l}{n-l}.$$

The above described summation spiral leads to

$$T'(n, m) = -2^{2m+1} n \binom{2n}{n} \sum_{i=0}^m \frac{2^{-2i} \binom{2i}{i}}{i+n} + 2 \binom{2m}{m} \binom{2n}{n} + 2^{2m+2n}.$$

If this relation is substituted in (3.9), then we arrive exactly at the assertion of Lemma 1.

Clearly, the case where  $m \geq n$  can be treated in a similar fashion. This finishes the algorithmic proof of Lemma 1.  $\square$

#### 4. AUXILIARY RESULTS

In this section, we show how to prove Lemmas 1–4 by making use of complex contour integrals. Before we can embark on the proofs of these lemmas, we need to establish several auxiliary evaluations of specific contour integrals.

**Lemma 6.** *For all non-negative integers  $n$ , we have*

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{dz}{z^{n+1}(1-z)^{n+1}} \frac{1}{(1-2z)} = 2^{2n}, \quad (4.1)$$

where  $\mathcal{C}$  is a contour close to 0, which encircles 0 once in the positive direction.

*Proof.* Let  $I_1$  denote the expression on the left-hand side of (4.1). We blow up the contour  $\mathcal{C}$  so that it is sent to infinity. While doing this, we must pass over the poles  $z = 1/2$  and  $z = 1$  of the integrand. This must be compensated by taking the residues at these points into account. Since the integrand is of the order  $O(z^{-2})$  as  $|z| \rightarrow \infty$ , the integral along the contour near infinity vanishes. Thus, we obtain

$$\begin{aligned} I_1 &= -\operatorname{Res}_{z=1/2} \frac{1}{z^{n+1}(1-z)^{n+1}} \frac{1}{(1-2z)} - \operatorname{Res}_{z=1} \frac{1}{z^{n+1}(1-z)^{n+1}} \frac{1}{(1-2z)} \\ &= 2^{2n+1} - \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{1}{(1+z)^{n+1}(1-(1+z))^{n+1}} \frac{1}{(1-2(1+z))} dz. \end{aligned}$$

As the substitution  $z \rightarrow -z$  shows, the last integral is identical with  $I_1$ . Thus, we have obtained an equation for  $I_1$ , from which we easily get the claimed result.  $\square$

**Lemma 7.** *For all non-negative integers  $n$ , we have*

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{dz}{z^{n+1}(1-z)^n} \frac{1}{(1-2z)} = 2^{2n-1} + \frac{1}{2} \binom{2n}{n}, \quad (4.2)$$

where  $\mathcal{C}$  is a contour close to 0, which encircles 0 once in the positive direction.

*Proof.* Denoting the expression on the left-hand side of (4.2) by  $I_2$ , we have

$$\begin{aligned} I_2 &= \frac{1}{2(2\pi i)} \int_{\mathcal{C}} \frac{dz}{z^{n+1}(1-z)^{n+1}} \frac{1}{(1-2z)} + \frac{1}{2(2\pi i)} \int_{\mathcal{C}} \frac{dz}{z^{n+1}(1-z)^{n+1}} \\ &= 2^{2n-1} + \frac{1}{2} \langle t^n \rangle (1-t)^{-n-1}, \end{aligned}$$

by Lemma 6. Upon coefficient extraction, this yields directly the right-hand side of (4.2).  $\square$

**Lemma 8.** *For all non-negative integers  $n$  and  $m$ , we have*

$$\frac{1}{(2\pi i)^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{1}{(u-t)} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} = -\frac{1}{2} \binom{2n+2m}{n+m}, \quad (4.3)$$

where  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are contours close to 0, which encircle 0 once in the positive direction, and  $\mathcal{C}_2$  is entirely in the interior of  $\mathcal{C}_1$ .

*Proof.* We treat here the case where  $n \geq m$ . The other case can be disposed of completely analogously.

Let  $I_4$  denote the expression on the left-hand side of (4.3). Clearly, interchange of  $u$  and  $t$  in the integrand does not change  $I_4$ . In that case however, we must also interchange the corresponding contours. Hence,  $I_4$  is also equal to one half of the sum of the original expression and the one where  $u$  and  $t$  are exchanged, that is,

$$\begin{aligned} I_4 &= \frac{1}{2(2\pi i)^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{1}{(u-t)} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \\ &\quad - \frac{1}{2(2\pi i)^2} \int_{\mathcal{C}_2} \int_{\mathcal{C}_1} \frac{1}{(u-t)} \frac{dt}{t^{n+1}(1-t)^{n+1}} \frac{du}{u^m(1-u)^m}. \end{aligned}$$

We would like to put both expressions under one integral. In order to do so, we must blow up the contour  $\mathcal{C}_2$  in the second integral (the contour for  $t$ ) so that it passes across  $\mathcal{C}_1$ . When doing so, the term  $u-t$  in the denominator will vanish, and so we shall collect a

residue at  $t = u$ . This yields

$$\begin{aligned}
I_4 &= \frac{1}{2(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{du dt}{(u-t)(u(1-u)t(1-t))^{n+1}} \left( (t(1-t))^{n-m+1} - (u(1-u))^{n-m+1} \right) \\
&\quad + \frac{1}{2(2\pi i)} \int_{C_1} \operatorname{Res}_{t=u} \frac{1}{(u-t)} \frac{dt}{t^{n+1}(1-t)^{n+1}} \frac{du}{u^m(1-u)^m} \\
&= \frac{1}{2(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{du dt (u+t-1)}{(u(1-u)t(1-t))^{n+1}} \sum_{\ell=0}^{n-m} (t(1-t))^\ell (u(1-u))^{n-m-\ell} \\
&\quad - \frac{1}{2(2\pi i)} \int_{C_1} \frac{du}{u^{n+m+1}(1-u)^{n+m+1}} \\
&= \sum_{\ell=0}^{n-m} \frac{1}{2(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{du dt}{u^{m+\ell}(1-u)^{m+\ell+1} (t(1-t))^{n-\ell+1}} \\
&\quad - \sum_{\ell=0}^{n-m} \frac{1}{2(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{du dt}{(u(1-u))^{m+\ell+1} t^{n-\ell+1} (1-t)^{n-\ell}} - \frac{1}{2} \binom{2n+2m}{n+m} \\
&= \frac{1}{2} \sum_{\ell=0}^{n-m} \binom{2n-2\ell}{n-\ell} \binom{2m+2\ell-1}{m+\ell} - \frac{1}{2} \sum_{\ell=0}^{n-m} \binom{2n-2\ell-1}{n-\ell-1} \binom{2m+2\ell}{m+\ell} \\
&\quad - \frac{1}{2} \binom{2n+2m}{n+m} = -\frac{1}{2} \binom{2n+2m}{n+m},
\end{aligned}$$

the last equality following from  $\binom{2k}{k} = 2\binom{2k-1}{k}$ .  $\square$

**Lemma 9.** *For all non-negative integers  $n$  and  $m$  with  $n \geq m$ , we have*

$$\begin{aligned}
&\frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{1}{(u-t)(1-2t)} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \\
&= -\frac{1}{4} \sum_{\ell=0}^{n-m} \binom{2n-2\ell}{n-\ell} \binom{2m+2\ell}{m+\ell} - 3 \cdot 2^{2n+2m-2}, \quad (4.4)
\end{aligned}$$

where  $C_1$  and  $C_2$  are contours close to 0, which encircle 0 once in the positive direction, and  $C_2$  is entirely in the interior of  $C_1$ . The sum on the right-hand side must be interpreted according to

$$\sum_{k=M}^{N-1} \operatorname{Expr}(k) = \begin{cases} \sum_{k=M}^{N-1} \operatorname{Expr}(k), & N > M, \\ 0, & N = M, \\ -\sum_{k=N}^{M-1} \operatorname{Expr}(k), & N < M. \end{cases} \quad (4.5)$$

*Proof.* Again, here we treat the case where  $n \geq m$ . The other case can be disposed of completely analogously.

Let  $I_5$  denote the expression on the left-hand side of (4.4). We apply the same trick as in the proof of Lemma 8 and observe that  $I_5$  is equal to one half of the sum of the original

expression and the one where  $u$  and  $t$  are exchanged, plus the residue of the latter at  $t = u$ . To be precise,

$$\begin{aligned}
I_5 &= \frac{1}{2(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{du dt}{(u-t)(1-2u)(1-2t)(u(1-u)t(1-t))^{n+1}} \\
&\quad \cdot \left( (1-2u)(t(1-t))^{n-m+1} - (1-2t)(u(1-u))^{n-m+1} \right) \\
&\quad + \frac{1}{2(2\pi i)^2} \int_{C_1} \text{Res}_{t=u} \frac{1}{(u-t)(1-2u)} \frac{1}{t^{n+1}(1-t)^{n+1}} \frac{du}{u^m(1-u)^m} \\
&= \frac{1}{2(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{du dt}{(u-t)(1-2t)(u(1-u)t(1-t))^{n+1}} \\
&\quad \cdot \left( (t(1-t))^{n-m+1} - (u(1-u))^{n-m+1} \right) \\
&\quad - \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{du dt}{(1-2u)(1-2t)(u(1-u))^m(t(1-t))^{n+1}} \\
&\quad - \frac{1}{2(2\pi i)^2} \int_{C_1} \frac{1}{(1-2u)} \frac{du}{u^{n+m+1}(1-u)^{n+m+1}} \\
&= \frac{1}{2(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{du dt (u+t-1)}{(1-2t)(u(1-u)t(1-t))^{n+1}} \sum_{\ell=0}^{n-m} (t(1-t))^\ell (u(1-u))^{n-m-\ell} \\
&\quad - 2^{2m-2+2n} - 2^{2n+2m-1} \\
&= \sum_{\ell=0}^{n-m} \frac{1}{2(2\pi i)^2} \int_{C'_1} \int_{C'_2} \frac{du dt}{(1-2t) u^{m+\ell}(1-u)^{m+\ell+1} (t(1-t))^{n-\ell+1}} \\
&\quad - \sum_{\ell=0}^{n-m} \frac{1}{2(2\pi i)^2} \int_{C'_1} \int_{C'_2} \frac{du dt}{(1-2t)(u(1-u))^{m+\ell+1} t^{n-\ell+1} (1-t)^{n-\ell}} - 3 \cdot 2^{2m+2n-2} \\
&= \frac{1}{2} \sum_{\ell=0}^{n-m} \binom{2m+2\ell-1}{m+\ell} 2^{2n-2\ell} \\
&\quad - \frac{1}{2} \sum_{\ell=0}^{n-m} \binom{2m+2\ell}{m+\ell} \left( 2^{2n-2\ell-1} + \frac{1}{2} \binom{2n-2\ell}{n-\ell} \right) - 3 \cdot 2^{2n+2m-2} \\
&= -\frac{1}{4} \sum_{\ell=0}^{n-m} \binom{2n-2\ell}{n-\ell} \binom{2m+2\ell}{m+\ell} - 3 \cdot 2^{2n+2m-2},
\end{aligned}$$

which is again seen by observing  $\binom{2k}{k} = 2\binom{2k-1}{k}$ . □

We are now in the position to prove the fundamental lemmas in Section 2.

*Proof of Lemma 1.* Using complex contour integrals, we may write

$$\begin{aligned} \sum_{0 \leq i \leq j} \binom{2n}{n+i} \binom{2m}{m+j} &= \sum_{0 \leq i \leq j} \binom{2n}{n-i} \binom{2m}{m-j} \\ &= \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{(1+x)^{2n}}{x^{n-i+1}} \frac{(1+y)^{2m}}{y^{m-j+1}} dx dy \\ &= \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{(1+x)^{2n}}{x^{n+1}} \frac{(1+y)^{2m}}{y^{m+1}} \frac{dx dy}{(1-xy)(1-y)}, \end{aligned}$$

where  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are contours close to 0, which encircle 0 once in the positive direction.

Now we do the substitutions  $x = u/(1-u)$  and  $y = t/(1-t)$ , implying  $dx = du/(1-u)^2$  and  $dy = dt/(1-t)^2$ . This leads to

$$\begin{aligned} \sum_{0 \leq i \leq j} \binom{2n}{n+i} \binom{2m}{m+j} &= \frac{1}{(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^{m+1}(1-t)^{m+1}} \frac{(1-u)(1-t)^2}{(1-u-t)(1-2t)} \\ &= \frac{1}{2(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^{m+1}(1-t)^{m+1}} \\ &\quad - \frac{1}{(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-2t)} \\ &\quad + \frac{1}{2(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^{m+1}(1-t)^{m+1}} \frac{1}{(1-2t)} \\ &\quad + \frac{1}{2(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)} \\ &\quad + \frac{1}{2(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)(1-2t)}. \end{aligned} \tag{4.6}$$

We now discuss the evaluation of the five integrals on the right-hand side one by one. First of all, we have

$$\begin{aligned} \frac{1}{2(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^{m+1}(1-t)^{m+1}} &= \frac{1}{2} \langle u^n \rangle (1-u)^{-n-1} \langle t^m \rangle (1-t)^{-m-1} \\ &= \frac{1}{2} \binom{2n}{n} \binom{2m}{m}. \end{aligned} \tag{4.7}$$

Next, by Lemma 6, we have

$$\frac{1}{(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-2t)} = 2^{2m-2} \binom{2n}{n} \tag{4.8}$$

and

$$\frac{1}{2(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^{m+1}(1-t)^{m+1}} \frac{1}{(1-2t)} = 2^{2m-1} \binom{2n}{n}. \quad (4.9)$$

In order to evaluate

$$I_6 := \frac{1}{2(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)},$$

we blow up the contour  $\mathcal{C}_1$  (the contour for  $u$ ) so that it is sent to infinity. While doing this, we pass over the poles  $u = 1-t$  and  $u = 1$  of the integrand. This must be compensated by taking the residues at these points into account. Since the integrand is of the order  $O(u^{-2})$  as  $|u| \rightarrow \infty$ , the integral along the contour near infinity vanishes. Thus, we obtain

$$\begin{aligned} I_6 &= -\frac{1}{2(2\pi i)} \int_{\mathcal{C}'_2} \text{Res}_{u=1-t} \frac{1}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)} \\ &\quad - \frac{1}{2(2\pi i)} \int_{\mathcal{C}'_2} \text{Res}_{u=1} \frac{1}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)} \\ &= \frac{1}{2(2\pi i)} \int_{\mathcal{C}'_2} \frac{dt}{t^{n+m+1}(1-t)^{n+m+1}} \\ &\quad - \frac{1}{2(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{(1+u)^{n+1}(1-(1+u))^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-(1+u)-t)} \\ &= \frac{1}{2} \binom{2n+2m}{n+m} - \frac{1}{4} \binom{2n+2m}{n+m} = \frac{1}{4} \binom{2n+2m}{n+m}, \end{aligned} \quad (4.10)$$

which is seen by performing the substitution  $u \rightarrow -u$  in the second expression in the next-to-last line and applying Lemma 8.

Finally, in order to evaluate

$$I_7 := \frac{1}{2(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)(1-2t)}$$

we again blow up the contour  $\mathcal{C}_1$  so that it is sent to infinity. While doing this, we pass over the poles  $u = 1-t$  and  $u = 1$  of the integrand. This must be compensated by taking the residues at these points into account. Since the integrand is of the order  $O(u^{-2})$  as

$|u| \rightarrow \infty$ , the integral along the contour near infinity vanishes. Thus, we obtain

$$\begin{aligned}
I_7 &= -\frac{1}{2(2\pi i)} \int_{\mathcal{C}'_2} \operatorname{Res}_{u=1-t} \frac{1}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)(1-2t)} \\
&\quad - \frac{1}{2(2\pi i)} \int_{\mathcal{C}'_2} \operatorname{Res}_{u=1} \frac{1}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)(1-2t)} \\
&= \frac{1}{2(2\pi i)} \int_{\mathcal{C}'_2} \frac{dt}{t^{n+m+1}(1-t)^{n+m+1}} \frac{1}{(1-2t)} \\
&\quad - \frac{1}{2(2\pi i)^2} \int_{\mathcal{C}'_2} \frac{du}{(1+u)^{n+1}(1-(1+u))^{n+1}} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-(1+u)-t)(1-2t)} \\
&= 2^{2n+2m-1} - \frac{1}{8} \sum_{\ell=0}^{n-m} \binom{2n-2\ell}{n-\ell} \binom{2m+2\ell}{m+\ell} - 3 \cdot 2^{2n+2m-3}, \tag{4.11}
\end{aligned}$$

which is seen by applying Lemma 6 to the first expression in the next-to-last line, performing the substitution  $u \rightarrow -u$  in the second expression, and applying Lemma 9. By combining (4.6)–(4.11) and simplifying, we obtain the right-hand side of (2.1).  $\square$

*Proof of Lemma 2.* We have

$$i \binom{2n}{n+i} = n \binom{2n}{n+i} - 2n \binom{2n-1}{n+i}.$$

Since we have already established Lemma 1, it remains to evaluate

$$\sum_{0 \leq i \leq j} \binom{2n-1}{n+i} \binom{2m}{m+j}.$$

Using complex contour integrals, we may write

$$\begin{aligned}
\sum_{0 \leq i \leq j} \binom{2n-1}{n+i} \binom{2m}{m+j} &= \sum_{0 \leq i \leq j} \binom{2n-1}{n-i-1} \binom{2m}{m-j} \\
&= \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{(1+x)^{2n-1}}{x^{n-i}} \frac{(1+y)^{2m}}{y^{m-j+1}} dx dy \\
&= \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{(1+x)^{2n-1}}{x^n} \frac{(1+y)^{2m}}{y^{m+1}} \frac{dx dy}{(1-xy)(1-y)},
\end{aligned}$$

where  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are contours close to 0, which encircle 0 once in the positive direction.



Now we do the substitutions  $x = u/(1-u)$  and  $y = t/(1-t)$ , implying  $dx = du/(1-u)^2$  and  $dy = dt/(1-t)^2$ . This leads to

$$\begin{aligned}
& \sum_{0 \leq i \leq j} \binom{2n-1}{n+i} \binom{2m}{m+j} \\
&= \frac{1}{(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{u^n(1-u)^{n+1}} \frac{dt}{t^{m+1}(1-t)^{m+1}} \frac{(1-u)(1-t)^2}{(1-u-t)(1-2t)} \\
&= \frac{1}{2(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{u^n(1-u)^n} \frac{dt}{t^{m+1}(1-t)^{m+1}} \frac{1}{(1-u-t)(1-2t)} \\
&\quad + \frac{1}{2(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{u^n(1-u)^n} \frac{dt}{t^{m+1}(1-t)^{m+1}} \frac{1}{(1-u-t)} \\
&\quad - \frac{1}{(2\pi i)^2} \int_{\mathcal{C}'_1} \int_{\mathcal{C}'_2} \frac{du}{u^n(1-u)^n} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)(1-2t)}.
\end{aligned}$$

The second expression on the right-hand side is an integral of the form  $I_6$  from the proof of Lemma 1, while the first and third expression are of the form  $I_7$  from the same proof.  $\square$

*Proof of Lemma 3.* We have

$$j \binom{2m}{m+j} = m \binom{2m}{m+j} - 2m \binom{2m-1}{m+j}.$$

Since we have already established Lemma 1, it remains to evaluate

$$\sum_{0 \leq i \leq j} \binom{2n}{n+i} \binom{2m-1}{m+j}.$$

Using complex contour integrals, we may write

$$\begin{aligned}
\sum_{0 \leq i \leq j} \binom{2n}{n+i} \binom{2m-1}{m+j} &= \sum_{0 \leq i \leq j} \binom{2n}{n-i} \binom{2m-1}{m-j-1} \\
&= \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{(1+x)^{2n}}{x^{n-i+1}} \frac{(1+y)^{2m-1}}{y^{m-j}} dx dy \\
&= \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{(1+x)^{2n}}{x^{n+1}} \frac{(1+y)^{2m-1}}{y^m} \frac{dx dy}{(1-xy)(1-y)},
\end{aligned}$$

where  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are contours close to 0, which encircle 0 once in the positive direction.

Now we do the substitutions  $x = u/(1-u)$  and  $y = t/(1-t)$ , implying  $dx = du/(1-u)^2$  and  $dy = dt/(1-t)^2$ . This leads to

$$\begin{aligned}
& \sum_{0 \leq i \leq j} \binom{2n}{n+i} \binom{2m-1}{m+j} \\
&= \frac{1}{(2\pi i)^2} \int_{C'_1} \int_{C'_2} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^{m+1}} \frac{(1-u)(1-t)^2}{(1-u-t)(1-2t)} \\
&= \frac{1}{(2\pi i)^2} \int_{C'_1} \int_{C'_2} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^m(1-t)^{m-1}} \frac{1}{(1-2t)} \\
&\quad + \frac{1}{(2\pi i)^2} \int_{C'_1} \int_{C'_2} \frac{du}{u^{n+1}(1-u)^{n+1}} \frac{dt}{t^{m-1}(1-t)^{m-1}} \frac{1}{(1-u-t)(1-2t)} \\
&= \left( 2^{2m-3} + \frac{1}{2} \binom{2m-2}{m-1} \right) \binom{2n}{n} + 2^{2n+2m-4} - \frac{1}{4} \sum_{\ell=0}^{n-m+1} \binom{2n-2\ell}{n-\ell} \binom{2m+2\ell-2}{m+\ell-1},
\end{aligned}$$

according to Lemma 6 and the evaluation of the expression  $I_7$  in the proof of Lemma 1.  $\square$

*Proof of Lemma 4.* We have

$$i j \binom{2n}{n+i} \binom{2m}{m+j} = \left( n \binom{2n}{n+i} - 2n \binom{2n-1}{n+i} \right) \left( m \binom{2m}{m+j} - 2m \binom{2m-1}{m+j} \right).$$

By Lemma 1, we have already evaluated the sum  $\sum_{0 \leq i \leq j} \binom{2n}{n+i} \binom{2m}{m+j}$ . Moreover, in the proofs of Lemmas 2 and 3 we evaluated the sums  $\sum_{0 \leq i \leq j} \binom{2n-1}{n+i} \binom{2m}{m+j}$  and  $\sum_{0 \leq i \leq j} \binom{2n}{n+i} \binom{2m-1}{m+j}$ . Hence, it remains to evaluate

$$\sum_{0 \leq i \leq j} \binom{2n-1}{n+i} \binom{2m-1}{m+j}.$$

Using complex contour integrals, we may write

$$\begin{aligned}
\sum_{0 \leq i \leq j} \binom{2n-1}{n+i} \binom{2m-1}{m+j} &= \sum_{0 \leq i \leq j} \binom{2n-1}{n-i-1} \binom{2m-1}{m-j-1} \\
&= \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{(1+x)^{2n-1}}{x^{n-i}} \frac{(1+y)^{2m-1}}{y^{m-j}} dx dy \\
&= \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{(1+x)^{2n-1}}{x^n} \frac{(1+y)^{2m-1}}{y^m} \frac{dx dy}{(1-xy)(1-y)},
\end{aligned}$$

where  $C_1$  and  $C_2$  are contours close to 0, which encircle 0 once in the positive direction.

Now we do the substitutions  $x = u/(1-u)$  and  $y = t/(1-t)$ , implying  $dx = du/(1-u)^2$  and  $dy = dt/(1-t)^2$ . This leads to

$$\begin{aligned} & \sum_{0 \leq i \leq j} \binom{2n-1}{n+i} \binom{2m-1}{m+j} \\ &= \frac{1}{(2\pi i)^2} \int_{C'_1} \int_{C'_2} \frac{du}{u^n(1-u)^{n+1}} \frac{dt}{t^m(1-t)^{m+1}} \frac{(1-u)(1-t)^2}{(1-u-t)(1-2t)} \\ &= \frac{1}{2(2\pi i)^2} \int_{C'_1} \int_{C'_2} \frac{du}{u^n(1-u)^n} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)} \\ & \quad + \frac{1}{2(2\pi i)^2} \int_{C'_1} \int_{C'_2} \frac{du}{u^n(1-u)^n} \frac{dt}{t^m(1-t)^m} \frac{1}{(1-u-t)(1-2t)}. \end{aligned}$$

The first expression on the right-hand side is an integral of the form  $I_6$  from the proof of Lemma 1, while the second expression is of the form  $I_7$  from the same proof.  $\square$

## 5. MAIN RESULT

This section contains our main result concerning double sums of the form

$$\sum_{0 \leq i \leq j \leq n} i^s j^t \binom{2n}{n+i} \binom{2n}{n+j}.$$

Aside from Lemmas 1–4, the proof of Proposition 10 below will also require a few more summation identities. These are given after the proof of Proposition 10 in Lemmas 11 and 12.

**Proposition 10.** *For all non-negative integers  $k, s, t$  and  $n$ , we have*

$$\begin{aligned} & \sum_{0 \leq i \leq j \leq n} i^s j^t \binom{2n}{n+i} \binom{2n}{n+j} = \frac{P_{s,t}^{(1)}(n)}{(4n-1)(4n-3) \cdots (4n-2S-2T+1)} \binom{4n}{2n} \\ & + \frac{P_{s,t}^{(2)}(n)}{(2n-1)(2n-3) \cdots (2n-2 \lfloor (S+T)/2 \rfloor + 1)} \binom{2n}{n}^2 + P_{s,t}^{(3)}(n) \cdot 4^n \binom{2n}{n} + P_{s,t}^{(4)}(n) \cdot 16^n, \end{aligned} \quad (5.1)$$

where  $P_{s,t}^{(i)}(n)$ ,  $i = 1, 2, 3, 4$ , are polynomials in  $n$ ,  $S = \lfloor s/2 \rfloor$  and  $T = \lfloor t/2 \rfloor$ . More specifically,

- (1) if  $s$  and  $t$  are even, then, as polynomials in  $n$ ,  $P_{s,t}^{(1)}(n)$  is of degree at most  $3S+3T$ ,  $P_{s,t}^{(2)}(n)$  is of degree at most  $2S+2T + \lfloor (S+T)/2 \rfloor$ ,  $P_{s,t}^{(3)}(n)$  is identically zero if  $s \neq 0$ ,  $P_{s,t}^{(3)}(n)$  is of degree at most  $2T$ , and  $P_{s,t}^{(4)}(n)$  is of degree at most  $2S+2T$ ;
- (2) if  $s$  is odd  $t$  is even, then, as polynomials in  $n$ ,  $P_{s,t}^{(1)}(n)$  is of degree at most  $3S+3T+1$ ,  $P_{s,t}^{(2)}(n)$  is of degree at most  $2S+2T+1 + \lfloor (S+T)/2 \rfloor$ ,  $P_{s,t}^{(3)}(n)$  is of degree at most  $2S+2T+1$ , and  $P_{s,t}^{(4)}(n)$  is identically zero;

- (3) if  $s$  is even and  $t$  is odd, then, as polynomials in  $n$ ,  $P_{s,t}^{(1)}(n)$  is of degree at most  $3S + 3T + 1$ ,  $P_{s,t}^{(2)}(n)$  is of degree at most  $2S + 2T + 1 + \lfloor (S + T)/2 \rfloor$ , and  $P_{s,t}^{(3)}(n)$  and  $P_{s,t}^{(4)}(n)$  are identically zero;
- (4) if  $s$  and  $t$  are odd, then, as polynomials in  $n$ ,  $P_{s,t}^{(1)}(n)$  is of degree at most  $3S + 3T + 2$ ,  $P_{s,t}^{(2)}(n)$  is of degree at most  $2S + 2T + 2 + \lfloor (S + T)/2 \rfloor$ , and  $P_{s,t}^{(3)}(n)$  and  $P_{s,t}^{(4)}(n)$  are identically zero.

*Proof.* We start with the case in which both  $s$  and  $t$  are even. With the notation of the theorem, we have  $s = 2S$  and  $t = 2T$ . We write

$$i^{2S} = \sum_{a=0}^S c_{a,S}(n) (n^2 - i^2) ((n-1)^2 - i^2) \cdots ((n-a+1)^2 - i^2), \quad (5.2)$$

where  $c_{a,S}(n)$  is a polynomial in  $n$  of degree  $2S - 2a$ ,  $a = 0, 1, \dots, S$ , and

$$j^{2T} = \sum_{b=0}^T c_{b,T}(n) (n^2 - j^2) ((n-1)^2 - j^2) \cdots ((n-b+1)^2 - j^2), \quad (5.3)$$

where  $c_{b,T}(n)$  is a polynomial in  $n$  of degree  $2T - 2b$ ,  $b = 0, 1, \dots, T$ . It should be noted that  $c_{S,S}(n) = c_{T,T}(n) = 1$ .

If we use the expansions (5.2) and (5.3) on the left-hand side of (5.1), then we obtain the expression

$$\begin{aligned} & \sum_{a=0}^S \sum_{b=0}^T c_{a,S}(n) c_{b,T}(n) \left( (2n - 2a + 1)_{2a} (2n - 2b + 1)_{2b} \sum_{0 \leq i \leq j} \binom{2n - 2a}{n + i - a} \binom{2n - 2b}{n + j - b} \right) \\ &= \sum_{a=0}^S \sum_{b=0}^T c_{a,S}(n) c_{b,T}(n) \left( (2n - 2a + 1)_{2a} (2n - 2b + 1)_{2b} \right. \\ & \quad \cdot \left( 2^{4n-2a-2b-3} + \frac{1}{4} \binom{4n-2a-2b}{2n-a-b} + \frac{1}{2} \binom{2n-2a}{n-a} \binom{2n-2b}{n-b} \right. \\ & \quad \left. \left. + 2^{2n-2b-2} \binom{2n-2a}{n-a} - \frac{1}{8} \sum_{\ell=0}^{b-a} \binom{2n-2a-2\ell}{n-a-\ell} \binom{2n-2b+2\ell}{n-b+\ell} \right) \right), \end{aligned}$$

due to Lemma 1 with  $n$  replaced by  $n - a$  and  $m = n - b$ . This expression can be further simplified by noting that

$$\sum_{a=0}^S c_{a,S}(n) (2n - 2a + 1)_{2a} \binom{2n-2a}{n-a} = 0^{2S} \binom{2n}{n},$$

which is equivalent to the expansion (5.2) for  $i = 0$ . Thus, we obtain

$$\begin{aligned} & \frac{1}{2} 0^{2S+2T} \binom{2n}{n}^2 + 0^{2S} \binom{2n}{n} \sum_{b=0}^T c_{b,T}(n) 2^{2n-2b-2} (2n-2b+1)_{2b} \\ & + \sum_{a=0}^S \sum_{b=0}^T c_{a,S}(n) c_{b,T}(n) \left( (2n-2a+1)_{2a} (2n-2b+1)_{2b} \right. \\ & \quad \cdot \left. \left( 2^{4n-2a-2b-3} + \frac{1}{4} \binom{4n-2a-2b}{2n-a-b} - \frac{1}{8} \sum_{\ell=0}^{b-a} \binom{2n-2a-2\ell}{n-a-\ell} \binom{2n-2b+2\ell}{n-b+\ell} \right) \right). \end{aligned}$$

Taking into account the properties of  $c_{a,S}(n)$  and  $c_{b,T}(n)$ , from this expression it is clear that  $P_{s,t}^{(4)}(n)$ , the coefficient of  $2^{4n} = 16^n$ , has degree at most  $2S + 2T$  as a polynomial in  $n$ . It is furthermore obvious that, due to the term  $0^{2S} = 0^s$ , the polynomial  $P_{s,t}^{(3)}(n)$ , the coefficient of  $2^{2n} \binom{2n}{n} = 4^n \binom{2n}{n}$ , vanishes for  $s \neq 0$ , while its degree is at most  $2T$  if  $s = 0$ .

In order to verify the claim about  $P_{s,t}^{(1)}(n)$ , the coefficient of  $\binom{4n}{2n}$ , we write

$$\begin{aligned} & c_{a,S}(n) c_{b,T}(n) (2n-2a+1)_{2a} (2n-2b+1)_{2b} \binom{4n-2a-2b}{2n-a-b} \\ & = c_{a,S}(n) c_{b,T}(n) \frac{(2n-2a+1)_{2a} (2n-2b+1)_{2b} (2n-a-b+1)_{a+b}^2}{(4n-2a-2b+1)_{2a+2b}} \binom{4n}{2n}. \end{aligned}$$

It is easy to see that  $(2n-a-b+1)_{a+b}$  divides numerator and denominator. After this division, the denominator becomes

$$2^{a+b} (4n-1)(4n-3) \cdots (4n-2a-2b+1),$$

that is, part of the denominator below  $P^{(1)}(n)$  in (5.1). The terms which are missing are

$$(4n-2a-2b-1)(4n-2a-2b-3) \cdots (4n-2S-2T+1).$$

Thus, if we put everything on the denominator

$$(4n-1)(4n-3) \cdots (4n-2S-2T+1),$$

then we see that the numerator of the coefficient of  $\binom{4n}{2n}$  has degree at most

$$(2S-2a) + (2T-2b) + 2a + 2b + 2(a+b) + (S+T-a-b) - (a+b) = 3S + 3T,$$

as desired.

Finally, we turn our attention to  $P_{s,t}^{(2)}(n)$ , the coefficient of  $\binom{2n}{n}^2$ . We have

$$c_{a,S}(n) c_{b,T}(n) (2n - 2a + 1)_{2a} (2n - 2b + 1)_{2b} \binom{2n - 2a - 2\ell}{n - a - \ell} \binom{2n - 2b + 2\ell}{n - b + \ell} \\ = c_{a,S}(n) c_{b,T}(n) \frac{(n - a - \ell + 1)_{a+\ell}^2 (n - b + \ell + 1)_{b-\ell}^2 (2n - 2b + 1)_{2\ell}}{(2n - 2a - 2\ell + 1)_{2\ell}} \binom{2n}{n}^2 \quad (5.4a)$$

$$= c_{a,S}(n) c_{b,T}(n) \frac{(n - a - \ell + 1)_{a+\ell}^2 (n - b + \ell + 1)_{b-\ell}^2 (2n - 2b + 1)_{2b-2a-2\ell}}{(2n - 2b + 2\ell + 1)_{2b-2a-2\ell}} \binom{2n}{n}^2. \quad (5.4b)$$

Let us assume  $a \leq b$ , in which case we need to consider non-negative indices  $\ell$ . (If  $a > b$ , then, according to the convention (4.5), we have to consider negative  $\ell$ . Using the definition (3.14) of the Pochhammer symbol for negative indices, the arguments would be completely analogous.) We make the further assumption that  $\ell \leq \frac{1}{2}(b - a)$  and use expression (5.4a). (If  $\ell > \frac{1}{2}(b - a)$ , then analogous arguments work starting from expression (5.4b).)

It is easy to see that  $(n - a - \ell + 1)_\ell$  divides numerator and denominator (as polynomials in  $n$ ) of the prefactor in (5.4a). Second, the (remaining) factor  $2^{2\ell}(n - a - \ell + \frac{1}{2})_\ell$  in the denominator and the factor  $(2n - 2b + 1)_{2\ell}$  in the numerator do not have common factors for  $\ell \leq \frac{1}{2}(b - a)$ . The denominator is a factor of the denominator below  $P_{s,t}^{(2)}(n)$  in (5.1). If in (5.4a) we extend denominator and numerator by the “missing” factor

$$(n - \lfloor (S + T)/2 \rfloor + \frac{1}{2})_{\lfloor (T+S)/2 \rfloor - \lfloor b+a \rfloor / 2} (n - a + \frac{1}{2})_a,$$

then, due to the properties of  $c_{a,S}(n)$  and  $c_{b,T}(n)$ , the numerator polynomial is of degree at most

$$(2S - 2a) + (2T - 2b) + 2(a + \ell) + 2(b - \ell) + 2\ell - \ell + \lfloor (T + S)/2 \rfloor - \lfloor (b + a)/2 \rfloor + a \\ = 2S + 2T + \ell + \lfloor (T + S)/2 \rfloor - \lfloor (b + a)/2 \rfloor + a \\ \leq 2S + 2T + \lfloor (b - a)/2 \rfloor + \lfloor (T + S)/2 \rfloor - \lfloor (b + a)/2 \rfloor + a \\ \leq 2S + 2T + \lfloor (S + T)/2 \rfloor,$$

as desired.

For the other cases, namely  $(s, t)$  being (odd, even), (even, odd), respectively (odd, odd), we proceed in the same way. That is, we apply the expansions (5.2) and (5.3) on the left-hand side of (5.1). Then, however, instead of Lemma 1, we apply Lemma 2, Lemma 3, and Lemma 4, respectively. The remaining arguments are completely analogous to those from the case of  $(s, t)$  being (even, even).  $\square$

**Lemma 11.** *For all non-negative integers  $n$  and  $k$ , we have*

$$\sum_{j=1}^n j^{2k} \binom{2n}{n+j} = -\frac{0^{2k}}{2} \binom{2n}{n} + 4^n \sum_{b=0}^k c_{b,k}(n) (2n - 2b + 1)_{2b} 2^{-2b-1}, \quad (5.5)$$

and

$$\sum_{j=1}^n j^{2k+1} \binom{2n}{n+j} = \frac{1}{2} \binom{2n}{n} \sum_{b=0}^k c_{b,k}(n) (n-b)_{b+1} (n-b+1)_b, \quad (5.6)$$

where the coefficients  $c_{b,k}(n)$  are defined in (5.3).

*Proof.* We use the expansion (5.3) with  $T = k$  on the left-hand side of (5.5). This gives

$$\begin{aligned} \sum_{j=1}^n j^{2k} \binom{2n}{n+j} &= \sum_{j=1}^n \sum_{b=0}^k c_{b,k}(n) (2n-2b+1)_{2b} \binom{2n-2b}{n+j-b} \\ &= \sum_{b=0}^k c_{b,k}(n) (2n-2b+1)_{2b} \left( 2^{2n-2b-1} - \frac{1}{2} \binom{2n-2b}{n-b} \right) \\ &= -\frac{0^{2k}}{2} \binom{2n}{n} + \sum_{b=0}^k c_{b,k}(n) (2n-2b+1)_{2b} 2^{2n-2b-1}, \end{aligned}$$

where we used (5.3) with  $T = k$  and  $j = 0$  in the last line. This is exactly the right-hand side of (5.5).

Now we do the same on the left-hand side of (5.6). This leads to

$$\begin{aligned} \sum_{j=1}^n j^{2k+1} \binom{2n}{n+j} &= \sum_{j=1}^n j \cdot \sum_{b=0}^k c_{b,k}(n) (2n-2b+1)_{2b} \binom{2n-2b}{n+j-b} \\ &= \sum_{b=0}^k c_{b,k}(n) (2n-2b)_{2b+1} \sum_{j=1}^n \left( \binom{2n-2b-1}{n+j-b-1} - \frac{1}{2} \binom{2n-2b}{n+j-b} \right) \\ &= \sum_{b=0}^k c_{b,k}(n) (2n-2b)_{2b+1} \left( 2^{2n-2b-2} - \frac{1}{2} 2^{2n-2b-1} + \frac{1}{4} \binom{2n-2b}{n-b} \right) \\ &= \frac{1}{2} \sum_{b=0}^k c_{b,k}(n) (n-b)_{b+1} (n-b+1)_b \binom{2n}{n}. \end{aligned}$$

This is exactly the right-hand side of (5.6).  $\square$

**Lemma 12.** For all non-negative integers  $n$  and  $k$ , we have

$$\sum_{j=1}^n j^{2k} \binom{2n}{n+j}^2 = -\frac{0^{2k}}{2} \binom{2n}{n}^2 + \frac{1}{2} \sum_{b=0}^k c_{b,k}(n) (2n-2b+1)_{2b} \binom{4n-2b}{2n-b} \quad (5.7)$$

and

$$\sum_{j=1}^n j^{2k+1} \binom{2n}{n+j}^2 = \sum_{b=0}^k c_{b,k}(n) (n-b)_{b+1} (n-b+1)_b \frac{n}{2(2n-b)} \binom{2n}{n}^2, \quad (5.8)$$

where the coefficients  $c_{b,k}(n)$  are defined in (5.3).

*Proof.* We start by using the expansion (5.3) with  $T = k$  on the left-hand side of (5.7). This gives

$$\sum_{j=1}^n j^{2k} \binom{2n}{n+j}^2 = \sum_{j=1}^n \sum_{b=0}^k c_{b,k}(n) (2n-2b+1)_{2b} \binom{2n-2b}{n+j-b} \binom{2n}{n+j}. \quad (5.9)$$

We have

$$\sum_{j=1}^n \binom{2n-2b}{n+j-b} \binom{2n}{n+j} = \sum_{j=-n}^{-1} \binom{2n-2b}{n+j-b} \binom{2n}{n+j}$$

and hence

$$\begin{aligned} \sum_{j=1}^n \binom{2n-2b}{n+j-b} \binom{2n}{n+j} &= -\frac{1}{2} \binom{2n-2b}{n-b} \binom{2n}{n} + \frac{1}{2} \sum_{j=-n}^n \binom{2n-2b}{n+j-b} \binom{2n}{n+j} \\ &= -\frac{1}{2} \binom{2n-2b}{n-b} \binom{2n}{n} + \frac{1}{2} \binom{4n-2b}{2n-b}, \end{aligned}$$

due to the Chu–Vandermonde summation. We substitute this back into (5.9) and obtain

$$\begin{aligned} \sum_{j=1}^n j^{2k} \binom{2n}{n+j}^2 &= \sum_{b=0}^k c_{b,k}(n) (2n-2b+1)_{2b} \left( -\frac{1}{2} \binom{2n-2b}{n-b} \binom{2n}{n} + \frac{1}{2} \binom{4n-2b}{2n-b} \right) \\ &= -\frac{0^{2k}}{2} \binom{2n}{n}^2 + \frac{1}{2} \sum_{b=0}^k c_{b,k}(n) (2n-2b+1)_{2b} \binom{4n-2b}{2n-b}, \end{aligned}$$

where we used (5.3) with  $T = k$  and  $j = 0$  in the last line.

In order to establish (5.8), we start again with (5.9), with an “additional”  $j$  on both sides,

$$\sum_{j=1}^n j^{2k+1} \binom{2n}{n+j}^2 = \sum_{j=1}^n j \cdot \sum_{b=0}^k c_{b,k}(n) (2n-2b+1)_{2b} \binom{2n-2b}{n+j-b} \binom{2n}{n+j}. \quad (5.10)$$

Using the standard hypergeometric notation

$${}_pF_q \left[ \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z \right] = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{m! (b_1)_m \cdots (b_q)_m} z^m,$$

where  $(\alpha)_m := \alpha(\alpha+1) \cdots (\alpha+m-1)$  for  $m \geq 1$ , and  $(\alpha)_0 := 1$ , we have

$$\sum_{j=1}^n j \binom{2n-2b}{n+j-b} \binom{2n}{n+j} = \binom{2n-2b}{n-b+1} \binom{2n}{n+1} {}_3F_2 \left[ \begin{matrix} 2, -n+1, -n+b+1 \\ n+2, n-b+2 \end{matrix}; 1 \right].$$

This  ${}_3F_2$ -series can be evaluated by means of (the terminating version) of Dixon’s summation (see [17, Appendix (III.9)])

$${}_3F_2 \left[ \begin{matrix} A, B, -N \\ 1+A-B, 1+A+N \end{matrix}; 1 \right] = \frac{(1+A)_N (1+\frac{A}{2}-B)_N}{(1+\frac{A}{2})_N (1+A-B)_N},$$



where  $N$  is a non-negative integer. Indeed, if we choose  $A = 2$ ,  $B = -n + 1$ , and  $N = n - b - 1$  in this summation formula, then we obtain

$$\sum_{j=1}^n j \binom{2n-2b}{n+j-b} \binom{2n}{n+j} = \frac{2n(n-b)}{2n-b} \binom{2n-2b-1}{n-b} \binom{2n-1}{n}.$$

If this is substituted back in (5.10), then we obtain the right-hand side of (5.8).  $\square$

## 6. SUMMATION FORMULAE FOR BINOMIAL DOUBLE SUMS INVOLVING ABSOLUTE VALUES

In this section we present the implications of Proposition 10 on sums of the form (1.2) with  $\beta = 1$ . As we point out in Remark 15(1) below, it would also be possible to derive similar theorems for arbitrary  $\beta$ .

**Theorem 13.** *For all non-negative integers  $k, s, t$  and  $n$ , we have*

$$\begin{aligned} & \sum_{-n \leq i, j \leq n} |i^s j^t (j^{2k} - i^{2k})| \binom{2n}{n+i} \binom{2n}{n+j} \\ &= \frac{U_k^{(1)}(n)}{(2n-1)(2n-3) \cdots (2n-2 \lfloor (S+T+k)/2 \rfloor + 1)} \binom{2n}{n}^2 + U_k^{(2)}(n) \cdot 4^n \binom{2n}{n}, \end{aligned} \quad (6.1)$$

where  $U_k^{(1)}(n)$  and  $U_k^{(2)}(n)$  are polynomials in  $n$ ,  $S = \lfloor s/2 \rfloor$  and  $T = \lfloor t/2 \rfloor$ .

More specifically,

- (1) if  $s$  and  $t$  are even, then, as polynomials in  $n$ ,  $U_{s,t}^{(1)}(n)$  is of degree at most  $2S + 2T + 2k + \lfloor (S+T+k)/2 \rfloor$ , and  $U_{s,t}^{(2)}(n)$  is identically zero;
- (2) if  $s$  is odd  $t$  is even, then, as polynomials in  $n$ ,  $U_{s,t}^{(1)}(n)$  is of degree at most  $2S + 2T + 2k + 1 + \lfloor (S+T+k)/2 \rfloor$ , and  $U_{s,t}^{(2)}(n)$  is of degree at most  $2S + 2T + 2k + 1$ ;
- (3) if  $s$  is even and  $t$  is odd, then, as polynomials in  $n$ ,  $U_{s,t}^{(1)}(n)$  is of degree at most  $2S + 2T + 2k + 1 + \lfloor (S+T+k)/2 \rfloor$ , and  $U_{s,t}^{(2)}(n)$  is of degree at most  $2S + 2T + 2k + 1$ ;
- (4) if  $s$  and  $t$  are odd, then, as polynomials in  $n$ ,  $U_{s,t}^{(1)}(n)$  is of degree at most  $2S + 2T + 2k + 2 + \lfloor (S+T+k)/2 \rfloor$ , and  $U_{s,t}^{(2)}(n)$  is identically zero.

*Proof.* The claim is trivially true for  $k = 0$ . Therefore we may assume from now on that  $k > 0$ .

Using the operations  $(i, j) \rightarrow (-i, j)$ ,  $(i, j) \rightarrow (i, -j)$ , and  $(i, j) \rightarrow (j, i)$ , which do not change the summand, we see that

$$\begin{aligned}
& \sum_{-n \leq i, j \leq n} |i^s j^t (j^{2k} - i^{2k})| \binom{2n}{n+i} \binom{2n}{n+j} \\
&= 4 \sum_{0 \leq i \leq j \leq n} \alpha(i=0) \alpha(j=0) (i^s j^t + i^t j^s) (j^{2k} - i^{2k}) \binom{2n}{n+i} \binom{2n}{n+j} \\
&= 4 \sum_{0 \leq i \leq j \leq n} (i^s j^t + i^t j^s) (j^{2k} - i^{2k}) \binom{2n}{n+i} \binom{2n}{n+j} \\
&\quad - 2 \binom{2n}{n} \sum_{j=1}^n (0^s j^t + 0^t j^s) j^{2k} \binom{2n}{n+j}, \tag{6.2}
\end{aligned}$$

where  $\alpha(\mathcal{A}) = \frac{1}{2}$  if  $\mathcal{A}$  is true and  $\alpha(\mathcal{A}) = 1$  otherwise. Now one splits the sums into several sums of the form

$$\sum_{0 \leq i \leq j \leq n} i^A j^B \binom{2n}{n+i} \binom{2n}{n+j}, \quad \text{respectively} \quad \sum_{j=1}^n j^B \binom{2n}{n+j}.$$

To sums of the second form, we apply Lemma 11, Eq. (5.5). In order to evaluate the sums of the first form, we proceed as in the proof of Proposition 10. That is, we apply the expansions (5.2) and (5.3) on the left-hand side of (5.1), and subsequently we use Lemmas 1–4 to evaluate the sums over  $i$  and  $j$ . Inspection of the result makes all assertions of the theorem obvious, except for the claims in Items (1) and (4) that the polynomial  $U_{s,t}^{(2)}(n)$ , the coefficient of  $4^n \binom{2n}{n}$  in (6.1), vanishes.

In order to verify these claims, we have to figure out what the coefficients of  $4^n \binom{2n}{n}$  of the various sums in (6.2) are precisely. For the case of even  $s$  and  $t$ , from Lemma 1 we

obtain that the coefficient of  $4^n \binom{2n}{n}$  in the expression (6.2) equals

$$\begin{aligned}
& 4 \sum_{a=0}^S \sum_{b=0}^{T+k} c_{a,S}(n) c_{b,T+k}(n) (2n-2a+1)_{2a} (2n-2b+1)_{2b} 2^{-2b-2} \binom{2n-2a}{n-a} \binom{2n}{n}^{-1} \\
& + 4 \sum_{a=0}^T \sum_{b=0}^{S+k} c_{a,T}(n) c_{b,S+k}(n) (2n-2a+1)_{2a} (2n-2b+1)_{2b} 2^{-2b-2} \binom{2n-2a}{n-a} \binom{2n}{n}^{-1} \\
& - 4 \sum_{a=0}^{S+k} \sum_{b=0}^T c_{a,S+k}(n) c_{b,T}(n) (2n-2a+1)_{2a} (2n-2b+1)_{2b} 2^{-2b-2} \binom{2n-2a}{n-a} \binom{2n}{n}^{-1} \\
& - 4 \sum_{a=0}^{T+k} \sum_{b=0}^S c_{a,T+k}(n) c_{b,S}(n) (2n-2a+1)_{2a} (2n-2b+1)_{2b} 2^{-2b-2} \binom{2n-2a}{n-a} \binom{2n}{n}^{-1} \\
& \quad - 2 \cdot 0^{2S} \sum_{b=0}^{T+k} c_{b,T+k}(n) (2n-2b+1)_{2b} 2^{-2b-1} \\
& \quad - 2 \cdot 0^{2T} \sum_{b=0}^{S+k} c_{b,S+k}(n) (2n-2b+1)_{2b} 2^{-2b-1} \\
& = 0^{2S} \sum_{b=0}^{T+k} c_{b,T+k}(n) (2n-2b+1)_{2b} 2^{-2b} + 0^{2T} \sum_{b=0}^{S+k} c_{b,S+k}(n) (2n-2b+1)_{2b} 2^{-2b} \\
& \quad - 0^{2S+2k} \sum_{b=0}^T c_{b,T}(n) (2n-2b+1)_{2b} 2^{-2b} - 0^{2T+2k} \sum_{b=0}^S c_{b,S}(n) (2n-2b+1)_{2b} 2^{-2b} \\
& \quad - 0^{2S} \sum_{b=0}^{T+k} c_{b,T+k}(n) (2n-2b+1)_{2b} 2^{-2b} - 0^{2T} \sum_{b=0}^{S+k} c_{b,S+k}(n) (2n-2b+1)_{2b} 2^{-2b},
\end{aligned}$$

which visibly vanishes due to our assumption that  $k > 0$ .

The proof for the analogous claim in Item (4) proceeds along the same lines. The only difference is that, instead of Lemma 1, here we need Lemma 4, and instead of (5.5) we need (5.6).  $\square$

**Theorem 14.** *For all non-negative integers  $k, s, t$  and  $n$ , we have*

$$\begin{aligned}
& \sum_{-n \leq i, j \leq n} |i^s j^t (j^{2k+1} - i^{2k+1})| \binom{2n}{n+i} \binom{2n}{n+j} \\
&= \frac{V_{s,t}^{(1)}(n)}{(4n-1)(4n-3) \cdots (4n-2S-2T-2k+1)} \binom{4n}{2n} \\
&+ \frac{V_{s,t}^{(2)}(n)}{(2n-1)(2n-3) \cdots (2n-2 \lfloor (S+T+k)/2 \rfloor + 1)} \binom{2n}{n}^2 + V_{s,t}^{(3)}(n) \cdot 4^n \binom{2n}{n} + V_{s,t}^{(4)}(n) \cdot 16^n,
\end{aligned} \tag{6.3}$$

where  $P_{s,t}^{(i)}(n)$ ,  $i = 1, 2, 3, 4$ , are polynomials in  $n$ ,  $S = \lfloor s/2 \rfloor$  and  $T = \lfloor t/2 \rfloor$ .

More specifically,

- (1) if  $s$  and  $t$  are even, then, as polynomials in  $n$ ,  $V_{s,t}^{(1)}(n)$  is of degree at most  $3S + 3T + 3k$ , and  $V_{s,t}^{(2)}(n)$ ,  $V_{s,t}^{(3)}(n)$ , and  $V_{s,t}^{(4)}(n)$  are identically zero;
- (2) if  $s$  is odd  $t$  is even, then, as polynomials in  $n$ ,  $V_{s,t}^{(2)}(n)$  is of degree at most  $2S + 2T + 2k + 1 + \lfloor (S+T+k)/2 \rfloor$ ,  $V_{s,t}^{(4)}(n)$  is of degree at most  $2S + 2T + 2k + 1$ , and  $V_{s,t}^{(1)}(n)$  and  $V_{s,t}^{(3)}(n)$  are identically zero;
- (3) if  $s$  is even and  $t$  is odd, then, as polynomials in  $n$ ,  $V_{s,t}^{(2)}(n)$  is of degree at most  $2S + 2T + 2k + 1 + \lfloor (S+T+k)/2 \rfloor$ ,  $V_{s,t}^{(4)}(n)$  is of degree at most  $2S + 2T + 2k + 1$ , and  $V_{s,t}^{(1)}(n)$  and  $V_{s,t}^{(3)}(n)$  are identically zero;
- (4) if  $s$  and  $t$  are odd, then, as polynomials in  $n$ ,  $V_{s,t}^{(1)}(n)$  is of degree at most  $3S + 3T + 3k + 2$ , and  $V_{s,t}^{(3)}(n)$  is of degree at most  $2S + 2T + 2k + 2$ , and  $V_{s,t}^{(2)}(n)$  and  $V_{s,t}^{(4)}(n)$  are identically zero.

*Proof.* We start again by using the operations  $(i, j) \rightarrow (-i, j)$ ,  $(i, j) \rightarrow (i, -j)$ , and  $(i, j) \rightarrow (j, i)$ . Here, they do change the summand. What we get is

$$\begin{aligned}
& \sum_{-n \leq i, j \leq n} |i^s j^t (j^{2k+1} - i^{2k+1})| \binom{2n}{n+i} \binom{2n}{n+j} \\
&= \frac{1}{4} \sum_{-n \leq i, j \leq n} \left( |i^s j^t (j^{2k+1} - i^{2k+1})| + |i^t j^s (j^{2k+1} - i^{2k+1})| \right. \\
&\quad \left. + |i^s j^t (j^{2k+1} + i^{2k+1})| + |i^t j^s (j^{2k+1} + i^{2k+1})| \right) \binom{2n}{n+i} \binom{2n}{n+j}
\end{aligned}$$

This symmetrised summand is now invariant under the above operations. Therefore, if we restrict the summation to the range  $0 \leq i \leq j \leq n$ , then this gives one eighth of the

complete sum, modulo some corrections of terms for which  $i = 0$ ,  $j = 0$ , or  $i = j$ ,

$$\begin{aligned}
& \sum_{-n \leq i, j \leq n} |i^s j^t (j^{2k+1} - i^{2k+1})| \binom{2n}{n+i} \binom{2n}{n+j} \\
&= 2 \sum_{0 \leq i \leq j \leq n} \alpha(i=0) \alpha(j=0) \alpha(i=j) \left( i^s j^{t+2k+1} - i^{s+2k+1} j^t + i^s j^{t+2k+1} + i^{s+2k+1} j^t \right. \\
&\quad \left. + i^t j^{s+2k+1} - i^{t+2k+1} j^s + i^t j^{s+2k+1} + i^{t+2k+1} j^s \right) \binom{2n}{n+i} \binom{2n}{n+j} \\
&= 4 \sum_{0 \leq i \leq j \leq n} \left( i^s j^{t+2k+1} + i^t j^{s+2k+1} \right) \binom{2n}{n+i} \binom{2n}{n+j} \\
&\quad - 2 \binom{2n}{n} \sum_{j=0}^n \left( 0^s j^{t+2k+1} + 0^t j^{s+2k+1} \right) \binom{2n}{n+j} \\
&\quad - 4 \sum_{j=0}^n j^{s+t+2k+1} \binom{2n}{n+j}^2,
\end{aligned}$$

where  $\alpha(\mathcal{A})$  has the same meaning as in the proof of Theorem 13.

From here on one proceeds in analogy with the arguments in the proof of Theorem 13. We leave the details to the reader.  $\square$

*Remark 15.* (1) It is obvious from the proofs of Theorems 13 and 14 that we could deduce analogous theorems for the more general sums (1.2). We omit this here for the sake of brevity.

(2) Theorems 13 and 14 imply an obvious algorithm to evaluate a sum of the form (1.2) for any given  $s, t, k$  and  $\beta = 1$ . (Again, an extension to arbitrary  $\beta$  would be possible.) Namely, addressing the case of odd  $k$ , one makes an indeterminate Ansatz for the polynomials  $V_{s,t}^{(1)}(n), V_{s,t}^{(2)}(n), V_{s,t}^{(3)}(n), V_{s,t}^{(4)}(n)$  in Theorem 14, one evaluates the sum on the left-hand side of (6.3) for  $n = 1, \dots, N$ , where  $N$  is the number of indeterminates involved in the Ansatz, giving rise to a system of  $N + 1$  linear equations for the  $N$  indeterminates. One solves the system and substitutes the solutions on the right-hand side of (6.3).

In this manner, we can establish any of the proved or conjectured double sum evaluations in [3]. For example, we obtain

$$\sum_{-n \leq i, j \leq n} |j^3 - i^3| \binom{2n}{n+i} \binom{2n}{n+j} = \frac{4n^2(5n-2)}{4n-1} \binom{4n-1}{2n-1}, \quad (6.4)$$

$$\sum_{-n \leq i, j \leq n} |j^5 - i^5| \binom{2n}{n+i} \binom{2n}{n+j} = \frac{8n^2(43n^3 - 70n^2 + 36n - 6)}{(4n-2)(4n-3)} \binom{4n-2}{2n-2}, \quad (6.5)$$

$$\sum_{-n \leq i, j \leq n} |ij(j^2 - i^2)| \binom{2n}{n+i} \binom{2n}{n+j} = \frac{2n^3(n-1)}{2n-1} \binom{2n}{n}^2, \quad (6.6)$$

$$\sum_{-n \leq i, j \leq n} |i^3 j^3(j^2 - i^2)| \binom{2n}{n+i} \binom{2n}{n+j} = \frac{2n^4(n-1)(3n^2 - 6n + 2)}{(2n-1)(2n-3)} \binom{2n}{n}^2, \quad (6.7)$$

$$\sum_{-n \leq i, j \leq n} |j^7 - i^7| \binom{2n}{n+i} \binom{2n}{n+j} = \frac{16n^2 P_1(n)}{(4n-3)(4n-4)(4n-5)} \binom{4n-3}{2n-3}, \quad (6.8)$$

where

$$P_1(n) = 531n^5 - 1960n^4 + 2800n^3 - 1952n^2 + 668n - 90,$$

all of which were conjectured in [3] (namely as (5.7)–(5.9), (5.12), (5.14)). However, our machinery also yields

$$\begin{aligned} & \sum_{-n \leq i, j \leq n} |i^4 j^3(j^5 - i^5)| \binom{2n}{n+i} \binom{2n}{n+j} \\ &= \frac{n^4(414n^6 - 2968n^5 + 8332n^4 - 11853n^3 + 9105n^2 - 3592n + 565)}{2(2n-5)(2n-3)(2n-1)} \binom{2n}{n}^2 \\ & \quad + \frac{1}{128} n^2(3n-1)(105n^3 - 210n^2 + 147n - 34) 16^n \end{aligned} \quad (6.9)$$

or

$$\begin{aligned} & \sum_{-n \leq i, j \leq n} |ij(j^3 - i^3)^3| \binom{2n}{n+i} \binom{2n}{n+j} \\ &= \frac{1}{16} n^2 (1377n^4 - 3870n^3 + 4503n^2 - 2442n + 496) 4^n \binom{2n}{n} \\ & \quad - \frac{4n^3 P_2(n)}{(4n-7)(4n-5)(4n-3)(4n-1)} \binom{4n}{2n}, \end{aligned} \quad (6.10)$$

where

$$P_2(n) = 1917n^7 - 11160n^6 + 26439n^5 - 33189n^4 + 23945n^3 - 9951n^2 + 2206n - 201,$$

for example.

## 7. AN INEQUALITY FOR A BINOMIAL DOUBLE SUM

In this final section, we establish Conjecture 3.1 from [3], which provides a lower bound on sums of the form (1.2), where the binomial coefficient  $\binom{2n}{n+i}$  is replaced by  $\binom{2m}{m+i}$ , and  $s = t = 0$ ,  $k = 2$ ,  $\beta = 1$ .

**Theorem 16.** *For all non-negative integers  $m$  and  $n$ , we have*

$$\sum_{i,j} |j^2 - i^2| \binom{2n}{n+i} \binom{2m}{m+j} \geq 2nm \binom{2n}{n} \binom{2m}{m}, \quad (7.1)$$

and equality holds if and only if  $m = n$ .

*Proof.* Without loss of generality, we assume  $m \geq n$ .

Using the operations  $(i, j) \rightarrow (-i, j)$  and  $(i, j) \rightarrow (i, -j)$ , which do not change the summand, we see that (7.1) is equivalent to

$$\sum_{0 \leq i, j} \alpha(i=0) \alpha(j=0) |j^2 - i^2| \binom{2n}{n+i} \binom{2m}{m+j} \geq \frac{nm}{2} \binom{2n}{n} \binom{2m}{m}, \quad (7.2)$$

where  $\alpha(i=0)$  has the same meaning as in the proof of Proposition 13. By Lemma 17, we see that the claim would be established if we were able to show that

$$\sum_{0 \leq i < j} \alpha(i=0) \left( \binom{2n}{n+i} \binom{2m-2}{m+j-1} - \binom{2n-2}{n+j-1} \binom{2m}{m+i} \right) \quad (7.3)$$

is non-negative, with equality holding only if  $m = n$ . Indeed, Lemma 19 says that these two last assertions hold even for each summand in (7.3) individually. (It is at this point that our assumption  $m \geq n$  comes into play.) This completes the proof of the theorem.  $\square$

**Lemma 17.** *For all non-negative integers  $m$  and  $n$ , we have*

$$\begin{aligned} & \sum_{0 \leq i, j} \alpha(i=0) \alpha(j=0) |j^2 - i^2| \binom{2n}{n+i} \binom{2m}{m+j} \\ &= \frac{nm}{2} \binom{2n}{n} \binom{2m}{m} \\ &+ 2(m-n) \sum_{0 \leq i < j} \alpha(i=0) \left( \binom{2n}{n+i} \binom{2m-2}{m+j-1} - \binom{2n-2}{n+j-1} \binom{2m}{m+i} \right). \end{aligned} \quad (7.4)$$

*Proof.* We write

$$j^2 - i^2 = (n^2 - i^2) - (m^2 - j^2) + (m^2 - n^2)$$

and decompose the sum on the left-hand side of (7.4) into two parts according to whether  $i < j$  or  $i > j$ . Thereby, the sum on the left-hand side of (7.4) becomes

$$\begin{aligned}
& (2n-1)_2 \sum_{0 \leq i < j} \alpha(i=0) \binom{2n-2}{n+i-1} \binom{2m}{m+j} \\
& \quad - (2m-1)_2 \sum_{0 \leq i < j} \alpha(i=0) \binom{2n}{n+i} \binom{2m-2}{m+j-1} \\
& \quad - (2n-1)_2 \sum_{0 \leq j < i} \alpha(j=0) \binom{2n-2}{n+i-1} \binom{2m}{m+j} \\
& \quad + (2m-1)_2 \sum_{0 \leq j < i} \alpha(j=0) \binom{2n}{n+i} \binom{2m-2}{m+j-1} \\
& \quad + (m^2 - n^2) \sum_{0 \leq i < j} \alpha(i=0) \left( \binom{2n}{n+i} \binom{2m}{m+j} - \binom{2n}{n+j} \binom{2m}{m+i} \right). \quad (7.5)
\end{aligned}$$

We next show how to evaluate the first two (double) sums in (7.5). In the first line of (7.5), we use the decomposition

$$\binom{2m}{m+j} = \binom{2m-2}{m+j} + 2 \binom{2m-2}{m+j-1} + \binom{2m-2}{m+j-2}, \quad (7.6)$$

while in the second line we use the same decomposition with  $m$  replaced by  $n$  and  $j$  by  $i$ . This leads to

$$\begin{aligned}
& (2n-1)_2 \sum_{0 \leq i < j} \alpha(i=0) \binom{2n-2}{n+i-1} \binom{2m}{m+j} \\
& \quad - (2m-1)_2 \sum_{0 \leq i < j} \alpha(i=0) \binom{2n}{n+i} \binom{2m-2}{m+j-1} \\
& = (2n-1)_2 \sum_{0 \leq i < j} \alpha(i=0) \binom{2n-2}{n+i-1} \binom{2m-2}{m+j} \\
& \quad + (2n-1)_2 \sum_{0 \leq i < j} \alpha(i=0) \binom{2n-2}{n+i-1} \binom{2m-2}{m+j-2} \\
& \quad - (2n-1)_2 \sum_{0 \leq i < j} \alpha(i=0) \binom{2n-2}{n+i} \binom{2m-2}{m+j-1} \\
& \quad - (2n-1)_2 \sum_{0 \leq i < j} \alpha(i=0) \binom{2n-2}{n+i-2} \binom{2m-2}{m+j-1} \\
& \quad + ((2n-1)_2 - (2m-1)_2) \sum_{0 \leq i < j} \alpha(i=0) \binom{2n}{n+i} \binom{2m-2}{m+j-1}.
\end{aligned}$$



By a simultaneous shift of  $i$  and  $j$  by one, one sees that the first and fourth sum on the right-hand side cancel each other largely, and the same is true for the second and the third sum. Thus, we have

$$\begin{aligned}
& (2n-1)_2 \sum_{0 \leq i < j} \alpha(i=0) \binom{2n-2}{n+i-1} \binom{2m}{m+j} \\
& - (2m-1)_2 \sum_{0 \leq i < j} \alpha(i=0) \binom{2n}{n+i} \binom{2m-2}{m+j-1} \\
& = -\frac{1}{2}(2n-1)_2 \sum_{0 < j} \binom{2n-2}{n-1} \binom{2m-2}{m+j} \\
& - \frac{1}{2}(2n-1)_2 \sum_{0 < j} \binom{2n-2}{n-2} \binom{2m-2}{m+j-1} \\
& + \frac{1}{2}(2n-1)_2 \sum_{0 < j} \binom{2n-2}{n-1} \binom{2m-2}{m+j-2} \\
& + \frac{1}{2}(2n-1)_2 \sum_{0 < j} \binom{2n-2}{n} \binom{2m-2}{m+j-1} \\
& + ((2n-1)_2 - (2m-1)_2) \sum_{0 \leq i < j} \alpha(i=0) \binom{2n}{n+i} \binom{2m-2}{m+j-1}.
\end{aligned}$$

Here, there is more cancellation: the second and fourth sum on the right-hand side cancel each other, while the first and third cancel each other in large parts, with only two terms remaining. As a result, we obtain

$$\begin{aligned}
& (2n-1)_2 \sum_{0 \leq i < j} \alpha(i=0) \binom{2n-2}{n+i-1} \binom{2m}{m+j} \\
& - (2m-1)_2 \sum_{0 \leq i < j} \alpha(i=0) \binom{2n}{n+i} \binom{2m-2}{m+j-1} \\
& = \frac{1}{2}(2n-1)_2 \binom{2n-2}{n-1} \binom{2m-1}{m} \\
& + ((2n-1)_2 - (2m-1)_2) \sum_{0 \leq i < j} \alpha(i=0) \binom{2n}{n+i} \binom{2m-2}{m+j-1} \\
& = \frac{n^2}{4} \binom{2n}{n} \binom{2m}{m} \\
& + ((2n-1)_2 - (2m-1)_2) \sum_{0 \leq i < j} \alpha(i=0) \binom{2n}{n+i} \binom{2m-2}{m+j-1}.
\end{aligned}$$

The same calculation, with  $n$  and  $m$  interchanged, yields

$$\begin{aligned}
& - (2n-1)_2 \sum_{0 \leq j < i} \alpha(j=0) \binom{2n-2}{n+i-1} \binom{2m}{m+j} \\
& + (2m-1)_2 \sum_{0 \leq j < i} \alpha(j=0) \binom{2n}{n+i} \binom{2m-2}{m+j-1} \\
& = \frac{m^2}{4} \binom{2n}{n} \binom{2m}{m} \\
& + ((2m-1)_2 - (2n-1)_2) \sum_{0 \leq i < j} \alpha(i=0) \binom{2m}{m+i} \binom{2n-2}{n+j-1}.
\end{aligned}$$

If we put everything together, then we have shown that the sum on the left-hand side of (7.4) equals

$$\begin{aligned}
& \frac{n^2 + m^2}{4} \binom{2n}{n} \binom{2m}{m} \\
& + (4(m^2 - n^2) - 2(m-n)) \\
& \times \sum_{0 \leq i < j} \alpha(i=0) \left( \binom{2n-2}{n+j-1} \binom{2m}{m+i} - \binom{2n}{n+i} \binom{2m-2}{m+j-1} \right) \\
& + (m^2 - n^2) \sum_{0 \leq i < j} \alpha(i=0) \left( \binom{2n}{n+i} \binom{2m}{m+j} - \binom{2n}{n+j} \binom{2m}{m+i} \right).
\end{aligned}$$

If we finally use Lemma 18 in this expression, then the result is the right-hand side of (7.4).  $\square$

**Lemma 18.** *For all non-negative integers  $m$  and  $n$ , we have*

$$\begin{aligned}
& 4 \sum_{0 \leq i < j} \alpha(i=0) \left( \binom{2n-2}{n+j-1} \binom{2m}{m+i} - \binom{2n}{n+i} \binom{2m-2}{m+j-1} \right) \\
& + \sum_{0 \leq i < j} \alpha(i=0) \left( \binom{2n}{n+i} \binom{2m}{m+j} - \binom{2n}{n+j} \binom{2m}{m+i} \right) \\
& = -\frac{m-n}{4(m+n)} \binom{2n}{n} \binom{2m}{m}. \quad (7.7)
\end{aligned}$$

*Proof.* Using the decomposition (7.6) in the second line of (7.7), we compute

$$\begin{aligned}
& 4 \sum_{0 \leq i < j} \alpha(i=0) \left( \binom{2n-2}{n+j-1} \binom{2m}{m+i} - \binom{2n}{n+i} \binom{2m-2}{m+j-1} \right) \\
& \quad + \sum_{0 \leq i < j} \alpha(i=0) \left( \binom{2n}{n+i} \binom{2m}{m+j} - \binom{2n}{n+j} \binom{2m}{m+i} \right) \\
& = \sum_{0 \leq i < j} \alpha(i=0) \left( 2 \binom{2n-2}{n+j-1} \binom{2m}{m+i} \right. \\
& \quad \left. - \binom{2n-2}{n+j} \binom{2m}{m+i} - \binom{2n-2}{n+j-2} \binom{2m}{m+i} \right. \\
& \quad \left. + \binom{2n}{n+i} \binom{2m-2}{m+j} + \binom{2n}{n+i} \binom{2m-2}{m+j-2} - 2 \binom{2n}{n+i} \binom{2m-2}{m+j-1} \right) \\
& = \sum_{0 \leq i} \alpha(i=0) \left( \binom{2n-2}{n+i} \binom{2m}{m+i} - \binom{2n-2}{n+i-1} \binom{2m}{m+i} \right. \\
& \quad \left. + \binom{2n}{n+i} \binom{2m-2}{m+i-1} - \binom{2n}{n+i} \binom{2m-2}{m+i} \right) \\
& = \frac{m-n}{m+n} \sum_{0 \leq i} \alpha(i=0) \left( \frac{(2n-2)!(2m-2)!(4nm-4(i+1)n-4(i+1)m+1)}{(n+i)!(n-i-1)!(m+i)!(m-i-1)!} \right. \\
& \quad \left. - \frac{(2n-2)!(2m-2)!(4nm-4in-4im+1)}{(n+i-1)!(n-i)!(m+i-1)!(m-i)!} \right) \\
& = \frac{m-n}{m+n} \left( -\frac{1}{2} \frac{(2n-2)!(2m-2)!(4nm-4n-4m+1)}{n!(n-1)!m!(m-1)!} \right. \\
& \quad \left. - \frac{1}{2} \frac{(2n-2)!(2m-2)!(4nm+1)}{(n-1)!n!(m-1)!m!} \right) \\
& = -\frac{m-n}{4(m+n)} \binom{2n}{n} \binom{2m}{m},
\end{aligned}$$

which is the desired result. Obviously, to obtain the telescoping form of the sum over  $i$ , we used Gosper's algorithm [7] (see also [12]). The particular implementation we used is the one due to Paule and Schorn [11].  $\square$

**Lemma 19.** *For all non-negative integers  $m, n, i, j$  with  $m \geq n$  and  $i < j$ , we have*

$$\binom{2n}{n+i} \binom{2m-2}{m+j-1} \geq \binom{2n-2}{n+j-1} \binom{2m}{m+i},$$

with equality if and only if  $m = n$ .

*Proof.* We have

$$\begin{aligned} \frac{\binom{2n}{n+i}\binom{2m-2}{m+j-1}}{\binom{2n-2}{n+j-1}\binom{2m}{m+i}} &= \frac{2n(2n-1)}{2m(2m-1)} \frac{(m-j+1)(m-j)}{(n-j+1)(n-j)} \prod_{k=i+1}^{j-1} \frac{(n+k)(m-k+1)}{(n-k+1)(m+k)} \\ &= \frac{\left(2 + \frac{2j-2}{n-j+1}\right) \left(2 + \frac{2j-1}{n-j}\right)}{\left(2 + \frac{2j-2}{m-j+1}\right) \left(2 + \frac{2j-1}{m-j}\right)} \prod_{k=i+1}^{j-1} \frac{nm + km - (k-1)n - k(k-1)}{nm - (k-1)m + kn - k(k-1)} \geq 1, \end{aligned}$$

and visibly equality holds if and only if  $m = n$ .  $\square$

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